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INSTABILITY OF STRATIFIED SHEAR FLOW

by

JÖRGEN HOLMBOE

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Department of Meteorology

University of California

Los Angeles

Contract No. AF19(604)-7999

Project No. 8628

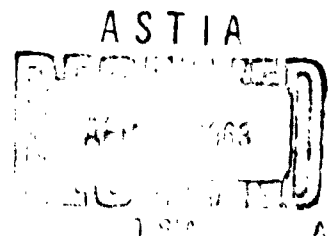
Task No. 86280

FINAL REPORT

March 1963

Prepared
for

GEOPHYSICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS



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INSTABILITY OF A STRATIFIED SHEAR FLOW

Introduction

This report contains a discussion of instability in a number of hydrodynamical models which have variations of density and velocity along the vertical. Most of the results which are presented here are not new. However, an attempt has been made to exhibit the similarity of the physical mechanism of the instability in the different models, which is sometimes hidden in the classical treatment of the subject. If the report has succeeded in this objective, it is at least in part because the so-called method of symmetric waves has been used in the mathematical analysis of the models. This method studies the instability as an initial value problem, instead of deriving the instability from the properties of the normal modes of the system.

The report may be regarded as a systematic introduction to the theory of long unstable cyclone waves in atmospheric models which take into account the compressibility of the air and the rotation of the earth. Much of the preliminary studies of these baroclinic waves has been carried out under the sponsorship of this contract, but is not yet ready for publication. This work will be continued and reported on under a current three year contract which is sponsored by the National Science Foundation.

March 1963

Jörgen Holmboe

INSTABILITY OF STRATIFIED SHEAR FLOW

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Shearing Instability

1. Instability of a vortex sheet. - Consider a system consisting of two unbounded homogeneous fluid layers with the same density which have the uniform relative translation $2U$ parallel to their plane surface of separation. This surface is called a vortex sheet. We shall use a frame of reference in which the fluid layers move in opposite directions with the same speed U . We call this the symmetric frame.

The vortex sheet may be regarded as a continuously distributed row of vortex filaments of equal strength and sense. The strength of the filaments in terms of the velocity discontinuity $2U$ is obtained by applying Stokes's theorem to a rectangular curve with one pair of sides parallel to the vortex sheet. If the length of these sides is L , the circulation of the curve is $2UL$, which is the combined strengths of the vortex filaments distributed along the length L of the vortex sheet. Since the distribution is uniform, the vortex strength per unit length of the vortex sheet is $2U$. This quantity is called the sliding vorticity of the vortex sheet.

It is readily seen that the vortex filaments are stationary in the symmetric frame. Consider any one of them. This filament is not moved by its own field. All the other filaments may be grouped in pairs whose neutral stagnation points coincide with the filament in question, so none of the fields move the filament. The same is true for all the vortex filaments. Accordingly the undisturbed plane vortex sheet is stationary.

Let us next assume that the vortex sheet is deformed into a sinusoidal surface of wave length $L = 2\pi/k$ and a small amplitude A_s , such that $kA_s \ll 1$.

Consider now the motion of the filaments. The filaments at the nodal points have no motion because all the other filaments, when grouped symmetrically in pairs with respect to a node, have fields whose neutral stagnation points coincide with the node. Consider next the filament on one of the crests. Again all the other filaments may be grouped in pairs symmetric with respect to the crest. The field from each pair gives the crest filament a partial motion in the direction of the flow in the upper layer. In the same way we find that the trough filaments move in the direction of the flow in the lower layer. This means that vorticity is being concentrated at the downwind nodes and depleted from the upwind nodes. It is readily seen that this periodic redistribution from the initial uniform distribution of the vortex filaments results in the evolution of a new field with ascending motion at the crests and descending motion in the troughs. So the amplitude of the vortex sheet deformation begins to grow.

The vertical motion associated with the periodic vorticity concentrations and depletions at the vortex sheet nodes has the same sense in both layers (see the middle diagram in fig. 1). This field adds to the initial vertical motion field which is associated with the deformed interface. Since the first streamlines in both layers coincide with the vortex sheet, the initial vertical motion is everywhere opposite in the two layers with maximum amplitude at the nodes. The resultant of this initial field and the evolving field from the periodic vorticity concentrations at the nodes is a field which moves downwind in both layers in the symmetric frame. During this evolution the deformation of the vortex sheet has no progressive motion in the symmetric frame, so at all times the flow in both layers is parallel to the vortex sheet at the nodes. This means that the amplitude of the vertical motion must grow faster than the amplitude of the vortex

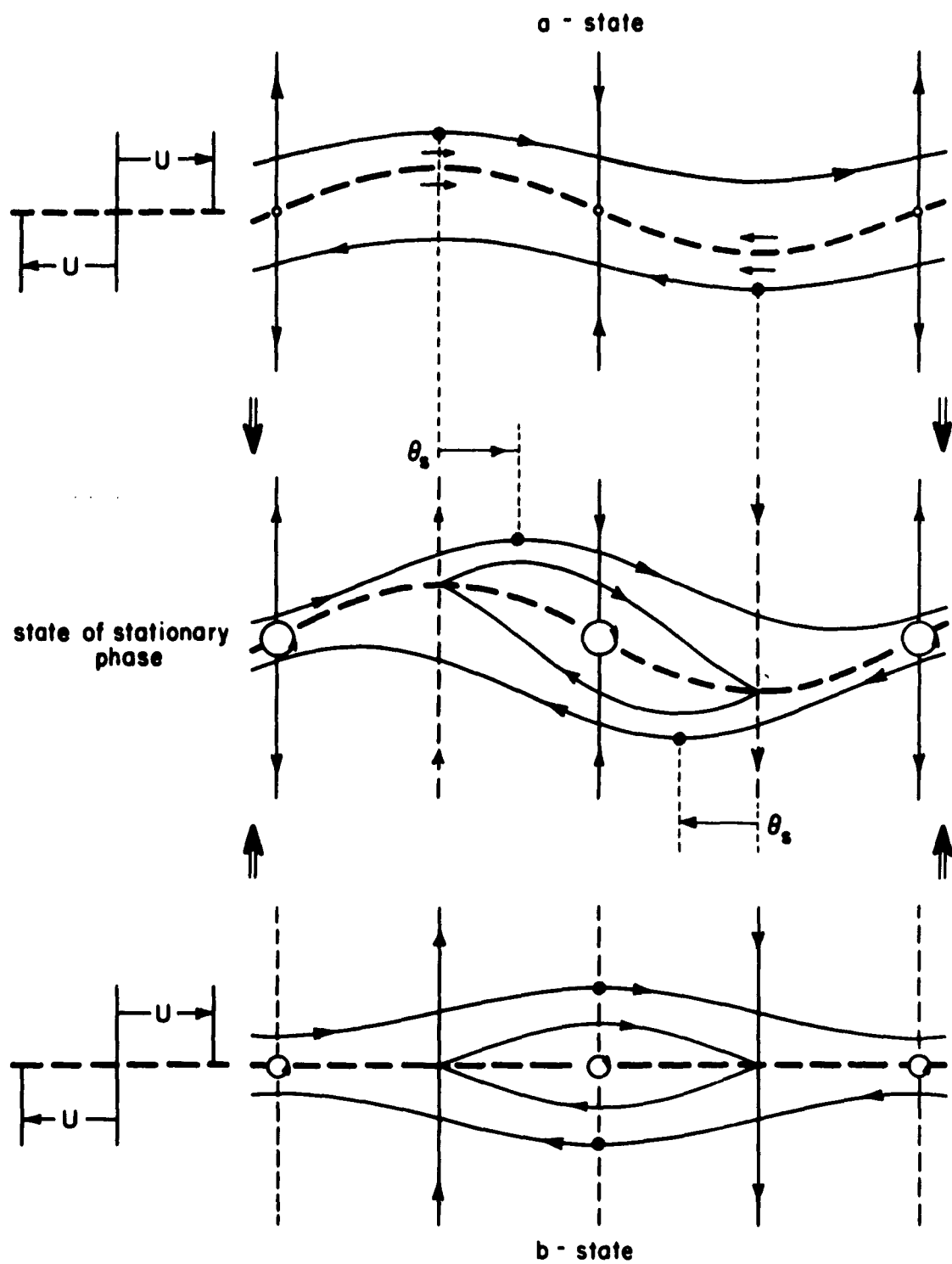


Fig. 1 Symmetric wave in a vortex sheet

sheet as long as the progressive motion of the wave continues.

The wave must stop before the upper and the lower field arrive in phase. For the streamlines in both layers are parallel to the vortex sheet at the nodes. If the upper and lower wave are placed in the same phase (lower diagram in fig. 1), the upper and lower streamlines are both horizontal at the points where the vortex sheet deformation should have its nodes. So the vortex sheet can have no deformation in this state. Since the vortex sheet amplitude must keep on growing after the initial state with the upper and the lower field in opposite phase (top diagram in fig. 1), it is evident that the wave cannot reach the state of equal phase which calls for no deformation of the interface. The wave must become stationary in some intermediate state (middle diagram in fig. 1). When this state of stationary phase is reached, both the wave amplitude and the vortex sheet amplitude will keep on growing at the same relative rate, as long as the simple sinusoidal periodicity is not destroyed, that is as long as the product kA_g remains much less than unity. During this early part of the evolution the growth of the vortex sheet amplitude is equal to the vertical velocity at the crest of the vortex sheet deformation. This vertical velocity is proportional to the vortex sheet amplitude, so $\dot{A}_g/A_g = \text{const.}$ In other words during the early linear part of the evolution the wave grows at an exponential rate in the state of stationary phase.

Consider the evolution of the wave from the state of equal phase (bottom diagram in fig. 1). This state has added vorticities distributed sinusoidally along the undeformed vortex sheet, with the maximum positive and negative vorticity added to the vortex filaments shown in the figure. The vertical component of this added field has the same phase in both layers. It deforms the vortex sheet into a sinusoidal surface with nodes

at the points of maximum added vorticity. As this deformation develops, it follows from the symmetry that it cannot have any progressive motion in the symmetric frame. So the streamlines are parallel to the vortex sheet at the nodes at all times. But this kinematic constraint forces both the upper and the lower field to move symmetrically upwind from the initial non-deformed state. Again it is clear that the wave in this evolution cannot reach the state of opposite phase since it moves downwind through that state.

The two non-tilting states shown at the bottom and the top in fig. 1 are accordingly mutually exclusive. In the evolution from either one of these states the wave cannot reach the other. From either non-tilting state the wave approaches the same state of stationary phase which has a downwind tilt from the non-tilting state with the deformed interface and an upwind tilt from the non-tilting state with the undeformed interface.

The wave in the vortex sheet which we have discussed here is called after its discoverer a symmetric Helmholtz wave. It is convenient to have short names for the two non-tilting states of the wave. The state with deformed vortex sheet and the upper and lower wave (vertical motion) in opposite phase is called the a-state. The state with a non-deformed vortex sheet and the upper and lower wave in phase is called the b-state. From either one of these the symmetric Helmholtz wave will approach asymptotically the same state of stationary phase with a downwind tilt from the a-state and exponential growth of the amplitudes as long as $A_g k \ll 1$.

The phase velocities with which the wave leaves the non-tilting states may be anticipated from dimensional considerations. The vortex sheet in the undisturbed state is defined by a single physical parameter namely the velocity U which measures one half of the sliding vorticity. When the wave is introduced the wave length is a second parameter. The

phase velocities in the non-tilting states must be related to these basic parameters, so they must both be proportional to U . The simplest choice suggests that the wave moves in opposite direction through the non-tilting states with the speed U and approaches a state of stationary phase half way between these states. The exponential growth rate in this asymptotic state has the dimension of inverse time, so it must be proportional to kU . It is in fact equal to kU .

The values of the stationary phase angle and the growth rate of the wave amplitude are obtained mathematically by applying the kinematic and dynamic boundary conditions to the vortex sheet: Let us first consider the kinematic conditions, which state that the fluid particles in the vortex sheet has the same vertical motion as the fluid particles immediately above and below, thus

$$(1.1) \quad \frac{Dz_s}{Dt} = w_1 = w_2. \quad (\text{Kinematic conditions})$$

Here z_s is the ordinate of the vortex sheet measured from its undeformed level ($z=0$), and the subscripts 1 and 2 refer to the upper and the lower layer. Since the motion in the layers is solenoidal, the velocity field \underline{v} of the wave disturbance may be represented by a streamfunction ψ whose sign we choose such that

$$(1.2) \quad \underline{v} = u\underline{i} + w\underline{k} = \nabla\psi \times \underline{j}.$$

Here \underline{i} , \underline{j} , \underline{k} are the orthogonal triple of unit vectors of a rectangular coordinate system (x,y,z) fixed to the symmetric frame with \underline{k} pointing upward from the lower to the upper layer, and \underline{i} pointing along the basic flow in the upper layer. \underline{j} points along the axis of the vortex filaments in the vortex sheet. The ordinate z is measured from the interface level.

Since the motion in the layers is irrotational, the streamfunction in (1.2) satisfies Laplace's equation. Since further the streamfunction has

a sinusoidal variation in the x-direction with the wave number k, its Laplace equation may be written

$$\nabla^2 \psi = \psi'' - k^2 \psi = 0.$$

The bounded solutions of this equation in the two layers are

$$(1.3) \quad \begin{aligned} \psi &= \psi_1 e^{-kz}, & (\text{in the upper layer}) \\ \psi &= \psi_2 e^{kz}. & (\text{in the lower layer}) \end{aligned}$$

Here ψ_1 and ψ_2 denote the streamfunction values at the interface level ($z=0$). These have the same amplitude in the symmetric wave, and by inspection of fig. 1 we see that the streamfunction in the upper layer has the same sign as z_s in the a-state. In an arbitrary tilting state of the wave with the phase θ measured positive downwind from the a-state the wave elements are therefore

$$(1.4) \quad \begin{aligned} z_s &= A_s \cos kx = \text{Real part } (A_s e^{ikx}) \\ \psi_1 &= UA \cos(kx - \theta) = \text{Real part } (Ze^{ikx}) \\ -\psi_2 &= UA \cos(kx + \theta) = \text{Real part } (Z^* e^{ikx}). \end{aligned}$$

On the right are introduced the complex wave parameter Z and its conjugate Z^* , defined by

$$(1.5) \quad Z = UAe^{-i\theta} = X - iY.$$

By including U in the amplitude factor UA of the streamfunction the amplitude A has the dimension of length. It is readily seen that A is the interface level streamline amplitude of the resultant field in the symmetric frame.

← Let z_1 denote the ordinate of this streamline in the upper layer.

The streamline slope at an arbitrary time, to the linear approximation, is

$$\frac{\partial z_1}{\partial x} = \frac{w_1}{U} = \frac{1}{U} \frac{\partial \psi_1}{\partial x}$$

which, using (1.4), gives

$$z_1 = A \cos(kx - \theta).$$

This is a sine curve with the amplitude A and the crest displaced the phase θ downwind from the interface crest as shown in the middle diagram, fig. 1. Since the interface deformation is stationary, the streamline is parallel to the interface at the node, so its amplitude is given by $A_s = A \cos \theta$. This condition is of course implicitly contained in the formal kinematic condition (1.1).

We note that the symmetric wave in (1.4), besides the two unchanging parameters U and k , is defined by the two amplitudes A_s , A and the phase θ . These three parameters are functions of time only. To find their evolution in time, we need three independent conditions. Two of these are the kinematic conditions in (1.1). The third is the dynamic condition that the vorticity of the filaments in the vortex sheet (the sliding vorticity) does not change with time.

It is convenient to represent these conditions non-dimensionally by choosing units of length and time such that $U=k=1$, so these parameters disappear from the equations. By this choice the growth rate of the wave amplitudes will be obtained non-dimensionally in units of kU .

Turning now to the kinematic condition (1.1) above the interface, we represent the individual change as the sum of local and convective changes in the symmetric frame and express the vertical velocity by the streamfunction in (1.2). With the non-linear convective term $\underline{v} \cdot \nabla \underline{z}_s$ ignored, the condition then takes the form

$$(1.6) \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) z_1 = w_1 = \frac{\partial \psi_1}{\partial x} .$$

If this condition is satisfied for the symmetric wave, the kinematic condition below the interface is automatically satisfied. One of the kinematic conditions in (1.1) is replaced by the condition of symmetry in (1.4), so

it is sufficient to consider (1.6). With the wave elements substituted here from (1.4), the exponential factor e^{ikx} cancels, and we get

$$\dot{A}_s + iA_s = iZ,$$

which, using (1.5) have the real and imaginary parts

$$\begin{aligned} \dot{A}_s &= Y = A \sin \theta \\ A_s &= X = A \cos \theta. \end{aligned} \quad \text{(Kinematic conditions).} \quad (1.7)$$

The first of these gives two equivalent expressions for the vertical velocity at the interface crest. The second condition states the fact, already noted earlier, that the streamline is parallel to the interface at the nodes. The ratio of the two kinematic conditions gives the growth rate of the interface amplitude in an arbitrary tilting state of the wave, namely

$$\frac{d}{dt} \ln A_s = \tan \theta. \quad (1.8)$$

The growth rate of the wave amplitude is obtained from the time derivative of the second condition combined with the first, namely

$$\dot{A}_s = \dot{A} \cos \theta - \dot{\theta} A \sin \theta = A \sin \theta.$$

Applied to the b-state ($\theta = 90^\circ$) this combined kinematic condition gives the phase velocity $\dot{\theta}_b = -1$. The wave moves upwind through the b-state with the speed U of the basic flow. In the b-state the propagation of the wave is governed by kinematic conditions only. The combined condition gives the growth rate of the wave amplitude in an arbitrary tilting state, namely

$$\frac{d}{dt} \ln A = (1 + \dot{\theta}) \tan \theta. \quad (1.9)$$

The wave is neutral in the non-tilting states. In a state of stationary phase ($\dot{\theta} = 0$) the wave and the interface deformation grow at the same rate, $n = \tan \theta_s$. In all other tilting states they grow at different rates.

We now turn to the dynamic boundary condition at the interface, which states that the surface vorticity of the vortex sheet filaments is con-

servative, that is

$$\frac{D}{Dt} (2U + u_1 - u_2) = 0.$$

With the individual change represented as the sum of the local and convective changes in the symmetric frame this condition, to the linear approximation, has the form

$$(1.10) \quad \frac{\partial}{\partial t} (u_1 - u_2) = -U \frac{\partial}{\partial x} (u_1 + u_2). \quad (\text{Dynamic condition}).$$

It is the mathematical formulation of the principle which we made use of in the qualitative discussion of the a-state at the beginning of this section, namely: The local increase of vorticity along the vortex sheet comes from the convergence of the surface vorticity. This interpretation of the dynamic boundary condition was given by Høiland* in 1942. The principle is most clearly exhibited in the a-state (top diagram in fig. 1) where the convergence along the vortex sheet has maximum strength at the downwind nodes.

With the horizontal velocities in (1.10) represented by the streamfunction in (1.2,3) the dynamic condition becomes

$$\frac{\partial}{\partial t} (\psi_1 + \psi_2) = -U \frac{\partial}{\partial x} (\psi_1 - \psi_2),$$

or with the wave elements of the symmetric wave in (1.4) substituted

$$(1.11) \quad \frac{d}{dt} (A \sin \theta) = A \cos \theta. \quad (Uk = 1).$$

This is the dynamic condition (conservation of sliding vorticity) for the symmetric wave in the vortex sheet. With the time differentiation carried out, it becomes

$$\dot{A} \sin \theta + \dot{\theta} A \cos \theta = A \cos \theta.$$

Applied to the a-state it gives the phase velocity of the wave through the a-state as $\dot{\theta}_a = 1$. The wave moves downwind through the a-state with the

* E. Høiland: The Developed Form of the Dynamic Boundary Condition with Applications, Arch. for Math. og Naturvid. B. XLVI. Nr. 2, 1942.

speed of the basic flow, so it has at that moment no intrinsic propagation through the fluid. The propagation through the a-state is determined by the dynamic condition only, namely by the vorticity concentrations at the downwind nodes, precisely as anticipated by qualitative reasoning earlier.

The dynamic condition gives another expression for the growth rate of the wave amplitude, namely

$$(1.12) \quad \begin{aligned} \frac{d}{dt} \ln A &= (1 - \dot{\theta}) \cot \theta, & (\text{Dynamic condition}) \\ \frac{d}{dt} \ln A &= (1 + \dot{\theta}) \tan \theta. & (\text{Kinematic condition}). \end{aligned}$$

The ratio of the two conditions gives the phase speed in an arbitrary tilting state, namely

$$(1.13) \quad \dot{\theta} = \cos^2 \theta - \sin^2 \theta = \cos 2\theta.$$

This formula verifies the earlier values in the non-tilting states, and it represents the phase velocity in an arbitrary tilting state as a linear combination of the values in the non-tilting states. In particular it shows that the wave from either non-tilting state approaches the state of stationary phase $\theta \rightarrow \theta_s = 45^\circ$, $\dot{\theta} \rightarrow 0$.

With the value of θ substituted, either equation (1.12) gives the explicit value of the growth rate of the wave amplitude in an arbitrary tilting state, namely

$$(1.14) \quad \frac{d}{dt} \ln A = \sin 2\theta = -\frac{1}{2} \frac{d}{dt} \ln \cos 2\theta.$$

By integration this equation gives the evolution of the wave from an arbitrary initial state with the phase θ_0 and the amplitude A_0 , namely

$$(1.15) \quad A^2 \cos 2\theta = A_0^2 \cos 2\theta_0 = A^2 \dot{\theta}.$$

A^2 is proportional to the kinetic wave energy, so θA^2 is the kinetic wave energy transport, which accordingly is constant during the evolution of the wave. As the wave slows down, the kinetic energy of the wave increases

a proportional amount. If the wave initially has a phase upwind from the a-state less than 45° the wave moves downwind toward the a-state. While approaching the a-state it speeds up and decays. It moves through the a-state with maximum speed (the speed of the fluid) and minimum amplitude. From the a-state the wave moves on downwind toward the state of stationary phase $\theta - \theta_s = 45^\circ$ and grows as it slows down. This evolution, which is expressed mathematically by the formula in (1.15), is represented graphically by a rectangular hyperbola in a polar diagram with A, θ as the polar coordinates. Each point on this hyperbola represents a state of the wave. During the evolution the "state-point" moves along the hyperbola toward increasing θ such that the radius vector moves with a constant areal velocity $\dot{\theta} A^2$. If the wave has a phase less than 45° downwind from the b-state it moves upwind from this state, and the evolution is represented graphically by the conjugate hyperbola which, during the evolution, is traced out in the sense of decreasing θ through the b-state and on toward asymptotic approach to the growing state of stationary phase.

We note from (1.13) that the wave has two states of stationary phase, namely the downwind state of $\theta_s = 45^\circ$ where the amplitude grows at the exponential rate $l = kU$, and the symmetrically located upwind state ($\theta_s = -45^\circ$) where the wave decays at the same rate. These two states of stationary phase are known as the normal modes of the vortex sheet. An arbitrary tilting state of the wave may be represented as the resultant of the two normal modes with the proper amplitude ratio. As one mode grows and the other decays the resultant tilting wave moves on toward the asymptotic growing state. An arbitrary tilting state of the wave may also be represented as the resultant of an a-state and a b-state with the proper amplitude ratio. As these two component waves move on toward their common state of stationary

phase in accordance with the evolution in (1.15), their resultant has the evolution of the resultant wave which again obeys (1.15). The pair of normal modes (the growing and decaying states of stationary phase) are thus closely related to the pair of non-tilting states of the symmetric wave. We shall consider this relationship further in the next section.

The evolution in (1.15) was derived from linearized theory which gives a valid approximation only as long as the amplitude is a small fraction of the wave length ($kA_s \ll 1$). Let us consider very briefly what happens after the linear approximation brakes down. We have seen (1.12) that the linear theory predicts the growth rate of the wave amplitude in the state of stationary phase as $l = kU = 2\pi/T$, where $T = L/U$ is the period during which the fluid moves a wave length in the symmetric frame. If in the state of stationary phase the wave at some initial time ($t = 0$) has the amplitude A_0 , its amplitude at the time t has grown to the value $A = A_0 \exp(2\pi t/T)$. The wave amplitude doubles during the time interval $t = 0.11T$, while the fluid moves a tenth of a wave length, so the motion appears to be highly unstable. However, this exponential growth will only continue as long as the amplitude is very much smaller than the wavelength, when the linearized theory gives a valid approximation. Later on, as the amplitude grows, the non-linear terms become important and can no longer be ignored. The non-linear evolution must be studied with the complete non-linear dynamic equations, which must be integrated numerically in short time steps. Such integrations were first carried out by Rosenhead* in 1931. The integration has been repeated more recently by several people with high speed electronic computers. These experiments have confirmed Rosenhead's earlier results. Instead of the continuous distribution of vortex filaments along the vortex sheet, Rosenhead

* The Formation of Vorticies from a Surface of Discontinuity. Proc. Roy. Soc. 1932, Series A, Vol. 134, p. 170-192.

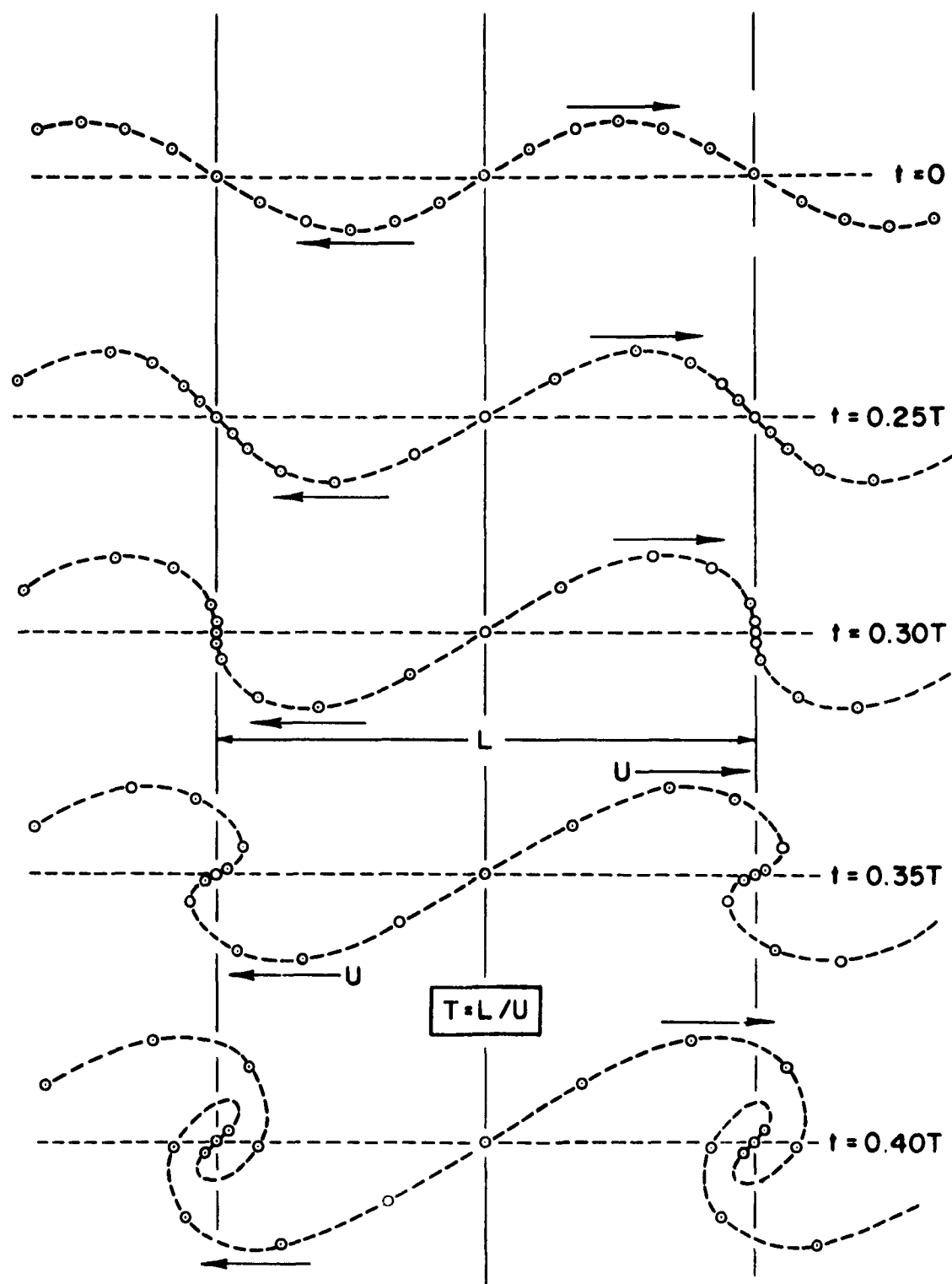


Fig. 1a Rolling up of a Vortex sheet (after Rosenhead)

carried out the computations with twelve vortex filaments uniformly distributed per wave length. His results are shown in fig. 1a. The concentration of the vortex filaments toward the downwind nodes, which was predicted by the linear theory, is clearly reflected by Rosenhead's discontinuous model during the evolution from the initial symmetric state at the top to the next state below a quarter period later. However, the amplitude grows at a much slower rate than the linear theory predicts. As the non-linear evolution continues, the growth of the amplitude soon stops, while the vorticity becomes increasingly concentrated on the steepening downwind slopes. The vorticity concentration is accompanied by a rolling up of the vortex sheet around the downwind nodes, a feature which was noted already by Helmholtz. The rolling up proceeds with increasing rapidity for the filaments near the nodal points.

There can be little doubt that a vortex sheet which is subjected to a small amplitude initial periodic disturbance, in the absence of viscous diffusion of the vorticity, behaves essentially in the way Rosenhead's computations predict: The wave crests at first grow almost symmetrically at the exponential rate predicted by the linear theory until their heights are about one tenth of the wave length. The subsequent non-linear evolution is asymmetric with gradual concentration of the sliding vorticity toward the nodal points on the steepening downwind slopes. As this process continues, the vortex sheet rolls up around the downwind nodes, and in the end the vortex sheet appears broken up into periodically spaced vortices with most of the vorticity confined within circular regions whose diameter is about four tenths of the wave length. The time required for this process from an initial deformation whose amplitude is one per cent of the wave length to the final vortex is roughly the time it takes the fluid in either

layer to move a wave length.

The vortex sheet is therefore an unstable hydrodynamical system. The smallest disturbance, whether periodic or not, will result in an evolution of the type described above during which the energy of the disturbance grows at the expense of the energy of the basic flow. This type of hydrodynamic instability is known as shearing instability. The instability will be partly suppressed if the fluid in the layers has different density with the lighter fluid on top, as will be shown in the next section. The vortex sheet between fluid layers of the same density exhibits the shearing instability in its pure form.

There is, however, a stabilizing effect in the vortex sheet which we have not considered, and which cannot be removed, namely the molecular viscous diffusion of vorticity. This diffusion process is going on all the time in real fluids in regions where there are vorticity gradients. This process is of course particularly active near the center of the rolled up vortex sheet, and tends to convert these regions into space vortices with continuous vorticity distribution. However, the diffusion tends to modify the vortex sheet itself even in the absence of any disturbance. This modification must be taken into account if we wish to predict the behavior of disturbances of a vortex sheet in a real fluid. This problem will be discussed in sections 6 and 7 below.

2. Symmetric waves in a statically stable vortex sheet. - Let now the unbounded homogeneous fluid layers with the relative translation $2U$ have different densities, with the lighter fluid ($\rho = \rho_1$) above and the heavier fluid ($\rho = \rho_2$) below the interface which separates the layers. The static stability of the interface is measured by the non-dimensional parameter

$$(2.1) \quad s = (\rho_0 - \rho_1)/(\rho_0 + \rho_1) \ll 1. \quad (\text{Quasi-symmetry condition}).$$

This system is not symmetric, for the lower layer has greater inertia.

However, if the density difference is a small fraction of the average density as indicated in (2.1), the asymmetry of the inertia can be ignored to a good approximation. The justification of this quasi-symmetry approximation is discussed in section 4.

If the symmetry approximation is justified, a symmetric wave with the wave elements in (1.4) will preserve its symmetry at all times. The evolutions of amplitudes and phase are governed by the two kinematic conditions in (1.7) and a dynamic condition which describes the changes of the sliding vorticity of the vortex filaments in the interface. These changes are most clearly visualized in the a-state of the wave (top diagram in fig. 1). The convergence along the interface concentrates the sliding vorticity toward the downwind nodes, as in the case of the statically neutral vortex sheet. However, the fluid filaments at these nodes are now light at the top and heavy at the bottom, and they are tilting downstream with the slope of the interface, so the upstream side is heavier than the downstream side. The gravitational buoyancy of this "over weight" will tend to return the nodal filaments to their equilibrium level positions, and thereby induce them to rotate in the sense opposite to the sliding vorticity. The resultant vorticity change is less than the partial change from either of these causes. If the convergence dominates, the wave moves downwind through the a-state with a slower speed than the fluid. The wave moves upwind through the b-state with unit speed. Therefore it has a state of stationary phase less than 45° downwind from the a-state, and its growth rate there, $n = \tan \theta_s$, is less than unity. If the overweight dominates, the wave moves upwind through the a-state and upwind through the b-state. It has no states of stationary phase.

It moves upwind at all times with rhythmic variations of phase speed and amplitude between the extreme values in the non-tilting states, while the energy transport in the symmetric frame, $\dot{\theta} A^2$, remains constant.

The vorticity change from the overweight must be proportional to the density difference and the force of gravity, that is $g(\rho_0 - \rho_1)$. It must therefore be proportional to the non-dimensional parameter

$$(2.2) \quad u = sg/kU^2 \quad (\text{Richardson number}),$$

where s is the non-dimensional static stability parameter in (2.1). The parameter u is called the Richardson number of the wave.

To find the proportionality factor, we consider the a -state of the wave for which the vorticity change from convergence is just balanced by the change from the overweight, so the resultant vorticity change at the nodes is zero. This wave is stationary in the a -state. It has a stationary neutral a -state with no local accelerations, so the convective accelerations are balanced by the acting forces (gravity and pressure force). But the convective accelerations are not changed if the direction of the flow is changed in one of the layers, say the lower layer (see top diagram in fig. 1). This new field is also steady. But this is the field of a gravity wave in a statically stable interface with no shear, in the frame which moves with the speed U of the wave. The speed of propagation of the gravity wave in the interface between unbounded layers is given by $U^2 = sg/k$, or $\mu = 1$. The wave mechanism of the gravity wave is very simple: Looking at the a -state diagram in fig. 1 with the direction of the flow reversed in the lower layer and inspecting the adjacent streamline channels above and below the interface, we note that the upper flow has maximum speed over the crest and minimum speed in the trough, while the speed below the interface changes in opposite rhythm. This means that the interface filament on the crest has maximum

positive vorticity and the filament in the trough has maximum negative vorticity. The interface is one of the streamlines, and the filaments move downwind along it with the speed U of the wave. On the journey from crest to trough the positive vorticity on the crest is changed to the negative vorticity in the trough. This change is caused by the overweight in the downstream hill between crest and trough. For a given overweight (density difference) the change proceeds at just the right rate when the filament moves along the streamline with the speed $U^2 = sg/k$.

The local vorticity change along a statically stable vortex sheet is accordingly given by the formula

$$\frac{\partial}{\partial t}(u_1 - u_2) = -(1 - \mu)U \frac{\partial}{\partial x}(u_1 + u_2). \quad (\text{Dynamic condition}).$$

The first term on the right represents the vorticity concentration by convergence (see 1.10), the μ -term gives the vorticity change from the overweight. The two effects act in opposite sense, and the ratio of their magnitudes is the Richardson number of the wave which is proportional to the wave length. The $(\mu=1)$ -wave has no local vorticity change. This wave has a stationary neutral a-state which is dynamically equivalent to the gravity wave we obtain by reversing the direction of the flow in one of the layers.

The dynamic condition has been anticipated here from the speed of the gravity wave. Because of its importance we shall now derive it from basic dynamic principles. Following Höiland's procedure (l.c.) it is obtained by integrating the equation of motion along a closed rectangular curve with horizontal and vertical sides. Its diagonal is an infinitesimal segment of the interface. The pressure is continuous across the interface (the dynamic condition) so the pressure integral is zero, and we get

$$\int \rho \left(\frac{D\mathbf{v}}{Dt} - \mathbf{g} \right) \cdot d\mathbf{r} = 0.$$

The rectangular curve has only four infinitesimal segments, one horizontal and one vertical in each layer. With second order (non-linear) terms ignored the integral becomes

$$(2.3) \quad \frac{D}{Dt} (\rho_1 u_1 - \rho_2 u_2) dx = g(\rho_2 - \rho_1) dz_s.$$

We now introduce the quasi-symmetry approximation by assuming that the difference of inertia between the layers can be ignored, and replace the densities ρ_1 and ρ_2 in the inertial terms on the left by the average density, $\frac{1}{2}(\rho_2 + \rho_1)$. The dynamic condition for the quasi-symmetric system then takes the simple form

$$(2.4) \quad \frac{D}{Dt} (u_1 - u_2) = 2sg \frac{\partial z_s}{\partial x}, \quad (\text{Dynamic condition}),$$

where s is the stability parameter in (2.1). This condition is intuitively clear: The sliding vorticity of the interface filament changes at a rate which is proportional to the stability and the slope of the interface. The lower half of the filament is heavy and the upper half is light, so the overweight tends to return the filament to its level position. With the individual changes in (2.4) represented as the sum of local and convective changes, the condition (to the linear approximation) takes the form

$$\frac{\partial}{\partial t} (u_1 - u_2) = -U \frac{\partial}{\partial x} (u_1 + u_2) + 2sg \frac{\partial z_s}{\partial x}. \quad (\text{Dynamic condition}).$$

Here the local change of the sliding vorticity along the interface appears as the resultant of two effects, namely: (i) the concentration of the sliding vorticity of the basic flow by convergence of the vortex filaments along the interface, and (ii) the action of the overweight of the filaments which tends to return them to their level positions. To compare the relative magnitude of the two effects we evaluate the interface slope from the difference of the kinematic conditions in (1.1), which gives

$$2U \frac{\partial z_s}{\partial x} = w_1 - w_2 = \frac{\partial}{\partial x} (\psi_1 - \psi_2) = \frac{1}{k} \frac{\partial}{\partial x} (u_1 + u_2).$$

With this value of the slope substituted, the dynamic condition takes the form which was anticipated from the dynamics of the gravity waves, namely

$$(2.5) \quad \frac{\partial}{\partial t} (u_1 - u_2) = -(1 - \mu)U \frac{\partial}{\partial x} (u_1 + u_2). \quad (\text{Dynamic condition})$$

Having derived this condition from basic principles, it now includes the theory of the gravity wave as equivalent to the a-state of the $(\mu=1)$ -wave, and predicts the speed of this wave as $U^2 = sg/k$.

With the wave elements of the symmetric wave in (1.4) substituted, the dynamic condition for this wave becomes

$$(2.6) \quad \frac{d}{dt} (A \sin \theta) = \dot{A} \sin \theta + \dot{\theta} A \cos \theta = (1 - \mu)A \cos \theta.$$

Applied to the a-state of the wave ($\theta=0$) it shows that the wave in that state has the speed $\dot{\theta}_a = 1 - \mu$. The short ($\mu < 1$)-waves move downstream, the longer ($\mu > 1$)-waves move upstream through the a-state. All waves move upstream through the b-state with the speed U of the flow, governed by the kinematic conditions which are independent of the static stability across the interface. So the short waves are unstable with a growing state of stationary phase, while the long waves are stable, moving upstream with rhythmic variations of phase speed and amplitude.

The details of the evolution are obtained by comparing the two equivalent expressions for the growth rate of the wave amplitude from the kinematic and dynamic conditions,

$$(2.7) \quad \frac{d}{dt} \ln A = (1 - \mu - \dot{\theta}) \cot \theta, \quad (\text{Dynamic condition})$$

$$\frac{d}{dt} \ln A = (1 + \dot{\theta}) \tan \theta. \quad (\text{Kinematic condition}).$$

The ratio of these give the phase speed in an arbitrary state as a linear combination of the non-tilting speeds

$$(2.8) \quad \dot{\theta} = (1 - \mu) \cos^2 \theta - \sin^2 \theta = \dot{\theta}_a \cos^2 \theta + \dot{\theta}_b \sin^2 \theta.$$

The short ($\mu < 1$)-waves have two states of stationary phase, namely

$$(2.9) \quad \tan^2 \theta_s = -\dot{\theta}_a / \dot{\theta}_b = 1 - \mu. \quad (\dot{\theta} = 0).$$

The dynamic and kinematic conditions give two equivalent values of the growth rate in these states, namely

$$(2.10) \quad \left(\frac{d}{dt} \ln A \right)_s = n = (1 - \mu) \cot \theta_s = \dot{\theta}_a \cot \theta_s$$

$$\left(\frac{d}{dt} \ln A \right)_s = n = \tan \theta_s = -\dot{\theta}_b \tan \theta_s$$

and their product determines the growth rate

$$(2.11) \quad n^2 = -\dot{\theta}_a \dot{\theta}_b = 1 - \mu = -m^2.$$

The growing state ($n > 0$) has the positive phase θ_s downstream from the a-state.

The decaying state ($n < 0$) has the same negative phase located symmetrically upstream from the a-state.

The long ($\mu > 1$)-waves move upstream with rhythmic variations of speed between the extreme values in the non-tilting states.

The evolution of the wave from an arbitrary tilting state is obtained by substituting the value of $\dot{\theta}$ in (2.8) into either one of the basic conditions (2.7), which gives

$$\frac{d}{dt} \ln A = (2 - \mu) \sin \theta \cos \theta = -\frac{1}{2} \frac{d}{dt} \ln \dot{\theta}.$$

The integral of this from an arbitrary initial state with the amplitude

A_0 and the phase θ_0 is

$$(2.12) \quad A^2 \dot{\theta} = A^2 [(1 - \mu) \cos^2 \theta - \sin^2 \theta] = A_0^2 [(1 - \mu) \cos^2 \theta_0 - \sin^2 \theta_0].$$

It shows that the energy transport of the wave is constant. If the initial state is an a-state with the amplitude $A_0 = A_a$, the equation becomes

$$(2.13) \quad (1 - \mu) \left(\frac{A \cos \theta}{A_a} \right)^2 - \left(\frac{A \sin \theta}{A_a} \right)^2 = 1 - \mu.$$

For the short ($\mu < 1$)-waves this is a hyperbola in a (A, θ) -polar diagram

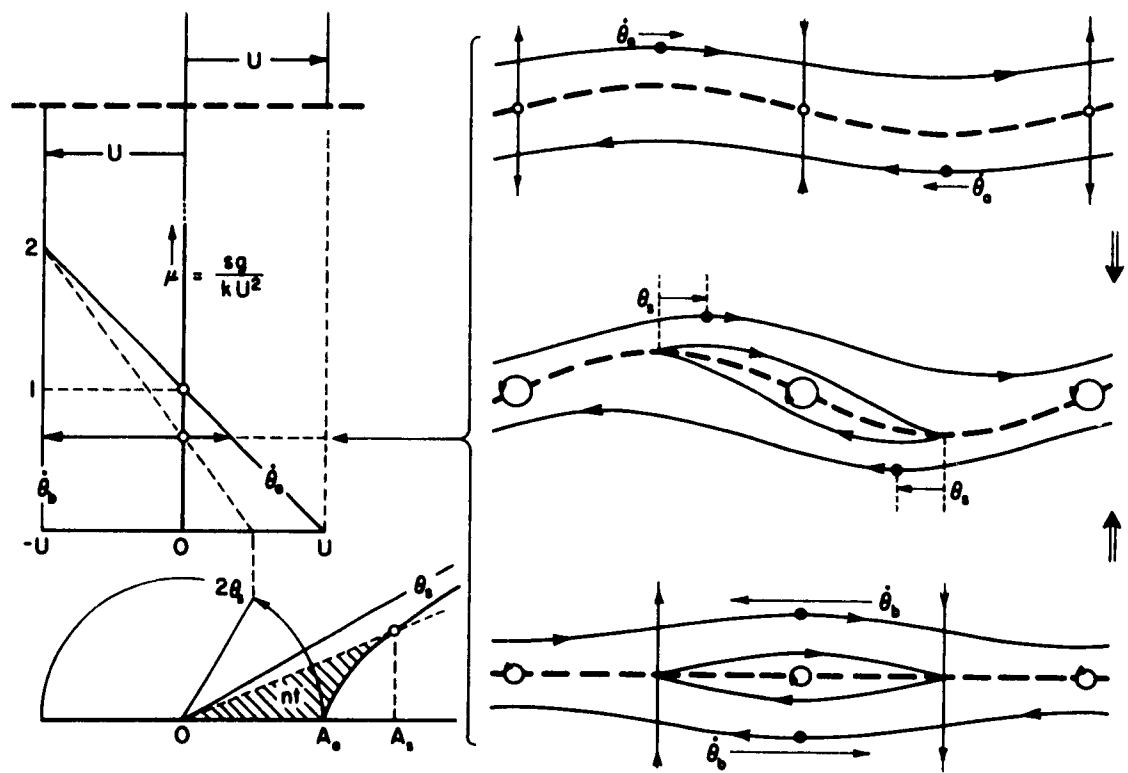


Fig. 2 Evolution of short unstable Helmholtz wave

with the asymptotic slope $\tan \theta_s = \sqrt{1-\mu} = n$. In the evolution from the a-state ($\theta=0$) the area between the hyperbola and the asymptote is swept out with the constant areal velocity

$$\frac{1}{2} A^2 \dot{\theta} = \frac{1}{2} A_a^2 \dot{\theta}_a = \frac{1}{2} (A_a^2 \tan \theta_s) n,$$

so the areal velocity is proportional to the asymptotic growth rate, as indicated in fig. 2. The evolution from an initial b-state ($\theta=90^\circ$) is represented by the corresponding conjugate hyperbola. The evolution from an arbitrary initial state (A_0, θ_0) is obtained by choosing the corresponding point on the hyperbola as the initial point.

For the long ($\mu > 1$)-waves equation (2.13) is an ellipse with the amplitude ratio $A_a/A_b = \sqrt{\mu-1} = m$, which is traced out in the upstream sense of decreasing θ during the evolution of the wave. The area of the ellipse is swept out at the constant rate

$$\frac{1}{2} \dot{\theta} A^2 = \frac{1}{2} \dot{\theta}_a A_a^2 = \frac{1}{2} \dot{\theta}_b A_b^2 = \frac{1}{2} m A_a A_b,$$

so the entire area of the ellipse, $\pi A_a A_b$, is swept over during the time $T = 2\pi/m$. This is the time during which the wave moves two full wavelengths while the interface performs a complete standing oscillation. So T is the period and m in (2.11) is the frequency of these stable oscillations. We note that the asymptotic growth rate of the unstable waves and the frequency of the stable waves are both equal to the geometric mean of the absolute values of the phase velocities in the non-tilting states, measured in the units of U and k .

All waves move upwind through the b-state with unit speed. The stable ($\mu < 2$)-waves move slower through the a-state so they decay during the passage from the a-state to the b-state. The longer ($\mu > 2$)-waves move with a speed greater than unity through the a-state, so these waves grow from the a-state to the b-state. The ($\mu = 2$)-wave is a neutral wave which moves upwind with

constant unit speed and constant amplitude.

3. Relation between the non-tilting states and the normal modes of the symmetric wave. - As mentioned in section 1, the states of stationary phase of the unstable symmetric waves are called the normal modes because the wave grows at a constant exponential rate. The statically stable quasi-symmetric vortex sheet between unbounded layers, defined by the parameters (U, s) , has a pair of unstable normal modes of this type for every wave length shorter than the stationary a-wave. For longer wave lengths the normal modes are a pair of neutral waves with the interface deformation propagating to the right or to the left with constant amplitude and the constant non-dimensional phase speed m in (2.11) relative to the symmetric frame. In the frame which moves with the mode the interface is a stationary streamline. The speeds of the flow in the layers relative to this stationary frame have the ratio $(1-m)/(1+m)$, and the amplitudes of the upper and lower part of the wave have evidently the same ratio. So the stable modes are not symmetric. However, the resultant of such a pair with the same amplitude is the stable symmetric wave with the standing oscillation of the interface and the upstream propagation of the wave as described at the end of section 2. The properties of the long stable $(\mu > 1)$ -modes are therefore known implicitly from the stable oscillations.

The boundary conditions (2.7) for the symmetric wave gave the relations (2.10) between the phase velocities in the non-tilting states and the growth rate and phase of the unstable modes, namely

$$(3.1) \quad \dot{\theta}_a = n \tan \theta_s, \quad \dot{\theta}_b = -n \cot \theta_s, \quad (n > 0)$$

where θ_s denotes the phase of the growing mode measured positive downwind from the a-state. These formulas are characteristic of all types of symmetric

waves, regardless of the physical mechanism which governs their propagation and growth. To show this, consider an arbitrary pair of unstable modes with the growth rates $\pm n$ and the phase $\pm \theta_s$. In each of these modes taken separately the upper field has the evolution

$$\begin{aligned} \psi_1^+ &= \frac{1}{2} e^{nt} \cos(x - \theta_s), \\ \psi_1^- &= \frac{1}{2} e^{-nt} \cos(x + \theta_s), \end{aligned} \quad (n > 0) \quad (3.2)$$

where an arbitrary amplitude factor is left out in each field. The resultant of these modes with equal initial amplitudes has the upper field

$$(3.3) \quad \psi_1 = chnt \cos \theta_s \cos x + shnt \sin \theta_s \sin x = A \cos(x - \theta).$$

This is a symmetric wave starting from an initial a-state ($\theta = t = 0$) with the amplitude $A_a = \cos \theta_s$, and the phase of the wave has the evolution

$$(3.4) \quad \tan \theta = \tan \theta_s th nt.$$

The wave moves toward the asymptotic state $\theta \rightarrow \theta_s$ as $t \rightarrow \infty$, with the phase speed

$$(3.5) \quad \dot{\theta} = n \tan \theta_s (\cos \theta / ch nt)^2 = n \tan \theta_s (A_a / A)^2,$$

in other words with the constant energy transport $\dot{\theta} A^2 = n \tan \theta_s A_a^2$. With $ch^{-2} nt = 1 - th^2 nt$ eliminated, using (3.4), the phase speed formula becomes

$$(3.6) \quad \dot{\theta} = n \tan \theta_s \cos^2 \theta - n \cot \theta_s \sin^2 \theta = \dot{\theta}_a \cos^2 \theta + \dot{\theta}_b \sin^2 \theta.$$

The result to the right is obtained by applying the formula to the non-tilting states ($\theta = 0, 90^\circ$). But these values of the phase speeds in the non-tilting states are the values in (3.1) which were obtained by examining the symmetric wave as an entity in its evolution from an arbitrary state. In other words the symmetric wave has the same evolution whether it is regarded as an entity or as a resultant of normal modes. This is a consequence of the linear theory. If several component waves satisfy the linearized boundary conditions, their resultant will also satisfy these conditions at all times. We may therefore use the spectrum of normal modes with their simple exponential

growth rates to examine the early linear stage in the evolution of arbitrary non-periodic local initial disturbances by representing these local disturbances as the resultant of the entire spectrum of symmetric waves with their amplitudes and phase properly adjusted to fit the local disturbance at the initial moment, using the general method of Fourier. The evolution of the local disturbance, for as long as the linear approximation is valid, is then obtained as the resultant of the separate evolutions of the component symmetric waves, which is known from section 2.

If the growth rate and stationary phase of the growing mode are known, the phase velocities in the non-tilting states are given by (3.1). Conversely if the phase velocities in the non-tilting states are known the growth rate and phase of the modes are given by the corresponding formulas

$$(3.7) \quad n^2 = -\dot{\theta}_a \dot{\theta}_b = -m^2; \quad \tan^2 \theta_s = -\dot{\theta}_a / \dot{\theta}_b.$$

These formulas define both the growing and the decaying mode. They have the same positive and negative growth rate and the same stationary phase downstream and upstream from the a-state. The sign of the phase of the growing mode ($n > 0$) is given by the sign of $\dot{\theta}_a$.

All the characteristics of the symmetric wave may then be obtained either from the growth rate and phase of the modes (n, θ_s) or from the phase velocities in the non-tilting states of the wave ($\dot{\theta}_a, \dot{\theta}_b$). Either pair of parameters may be derived independently of the other pair, which then in turn are obtained from the interrelated formulas in (3.1) or (3.7). The question of which pair should be selected for primary derivation depends upon the circumstances, and sometimes it is a matter of taste. The more mathematically inclined student would probably in most cases prefer to determine the normal mode pair (n, θ_s) first. The phase velocities in the non-tilting states ($\dot{\theta}_a, \dot{\theta}_b$) on the other hand are often accessible by a

simple physical consideration, and may sometimes be derived intuitively from fundamental physical principles with very little mathematical labor.

However, both sets of parameters are useful tools for the physical analysis of the wave mechanism: If the wave moves in opposite directions through the non-tilting states it starts growing when it leaves these states, and it approaches from either side the growing state of stationary phase whose growth rate and phase are uniquely determined by the non-tilting phase speeds. If the wave moves in the same direction through the non-tilting states, it is a progressive wave in that direction with rhythmic variations of speed and amplitude in such a way that the energy transport θA^2 remains constant. Meanwhile the interface performs standing oscillations, with the frequency equal to the geometric mean of the non-tilting phase speeds. Both growth rate and stationary phase are imaginary in the stable wave. But the formula for the field in (3.3) is real and describes the evolution of the stable wave from an initial a-state.

4. Justification of the symmetry approximation. - We shall now examine the magnitude of the error we make in the quasi-symmetric wave theory by ignoring the kinematic effect of the density difference in the dynamic boundary condition. To determine the error we must find the behavior of the waves in a system with arbitrary densities in the layers, and examine how much it departs from the behavior of the quasi-symmetric waves when the density difference is small.

To make the analysis a little more general we shall let the layers be bounded by rigid horizontal planes at equal distances above and below the interface, so both layers have the same finite depth h . This system has a new length parameter h , besides the wave number k , but the theory for the

waves in the bounded system must reduce to the wave theory for unbounded layers for the waves which are very much shorter than the depth of the layers ($kh \gg 1$). In fact, we see from (1.3) that the amplitude of the wave in the unbounded system decreases exponentially with the distance from the interface and is reduced to about 4 per cent of the central value at a distance of half a wave length from the interface. If the rigid boundaries are further away they have very little effect on the waves. However, if they are closer, the streamfunction of the wave must be such that the vertical motion $\partial\psi/\partial z$ is zero at the boundaries. Since the streamfunction is periodic in the horizontal direction, this means that it is zero at the outer rigid boundaries. The motion is of course still Laplacean, so the streamfunction must be a linear combination of the basic solutions in (1.3) which is zero at the height h above and at the depth h below the interface. In other words, the streamfunction of the wave in the bounded system must have the form

$$(4.1) \quad \begin{aligned} \psi &= \psi_1 \operatorname{sh} k(h-z)/\operatorname{sh} kh, & (\text{Upper layer}) \\ \psi &= \psi_2 \operatorname{sh} k(h+z)/\operatorname{sh} kh, & (\text{Lower layer}) \end{aligned}$$

where, as before, ψ_1 and ψ_2 are the values in the two layers at the interface level. The corresponding horizontal velocities are

$$(4.2) \quad \begin{aligned} u &= \psi_1 k \operatorname{ch} k(h-z)/\operatorname{sh} kh, & (\text{Upper layer}) \\ -u &= \psi_2 k \operatorname{ch} k(h+z)/\operatorname{sh} kh. & (\text{Lower layer}) \end{aligned}$$

We see that the wave has no vertical wind-shear near the rigid boundaries. This may be anticipated from the fact that the irrotational flow is here parallel to the boundaries, so both the curvature and shear are zero. We shall consider the consequences of this circumstance later.

If the lower fluid is much heavier than the upper fluid, the inertia of the system is highly asymmetric with respect to the interface. It is

therefore no reason to expect that an initially symmetric wave with the wave elements in (1.4) will remain symmetric during its evolution. Rather the contrary must be expected. We shall therefore investigate whether this asymmetric system has normal modes for which the interface deformation has the evolution

$$(4.3) \quad z_s = A_{so} \times \text{Real part of } \exp i(kx + mt).$$

The frequency equation, $m = m(k)$ for the mode is found by applying the two kinematic conditions (1.1) and the dynamic condition (2.3) to the interface deformation. If for a given value of k the frequency equation gives a real positive value for m , the mode is a stable neutral wave which moves in the symmetric frame toward negative x with the phase speed m/k . If the frequency equation gives an imaginary frequency $m = -in$, the mode is an unstable wave whose amplitude grows at the exponential rate n .

We shall again represent the boundary conditions non-dimensionally by using units of length and time such that $k = U = 1$. The local change in the symmetric frame of the deformation in (4.3) is then

$$\frac{\partial z_s}{\partial t} = im z_s = \frac{m}{k} \frac{\partial z_s}{\partial x} = m \frac{\partial z_s}{\partial x},$$

so the kinematic conditions (1.1) for the mode are

$$(4.4) \quad z_s = \psi_1 / (m + 1) = \psi_2 / (m - 1).$$

The dynamic condition in (2.3) is

$$\frac{D}{Dt} (\rho_1 u_1 - \rho_2 u_2) = g(\rho_2 - \rho_1) \frac{\partial z_s}{\partial x}.$$

From (4.2) the horizontal accelerations at the interface level are

$$\frac{Du_1}{Dt} = \text{cth } kh \frac{D\psi_1}{Dt}; \quad -\frac{Du_2}{Dt} = \text{cth } kh \frac{D\psi_2}{Dt}; \quad (k=1)$$

or with the streamfunction values of the mode substituted from (4.4)

$$\frac{Du_1}{Dt} = (m+1) \csc kh \frac{Dz_s}{Dt} = (m+1)^2 \csc kh \frac{\partial z_s}{\partial x},$$

$$-\frac{Du_2}{Dt} = (m-1) \csc kh \frac{Dz_s}{Dt} = (m-1)^2 \csc kh \frac{\partial z_s}{\partial x}.$$

With these substituted the dynamic condition becomes

$$\rho_1 (m+1)^2 + \rho_2 (m-1)^2 = g(\rho_2 - \rho_1) \csc kh.$$

This is the frequency equation for the normal modes (4.3) in the statically stable vortex sheet between bounded layers. The Richardson number of the waves in the bounded system is defined as

$$(4.5) \quad \mu = (sg/kU^2) \csc kh.$$

It reduces to the value in the unbounded system when the wave is very much shorter than the depth of the layers. With this value of μ substituted the frequency equation becomes

$$m^2 - 2sm = \mu - 1$$

or, with s^2 added on both sides,

$$(4.6) \quad (m-s)^2 = \mu - (1-s^2).$$

If the density difference is a small fraction of the mean density ($s \ll 1$) the frequencies of the normal modes are obtained with good approximation by ignoring s in (4.6), giving $m^2 = \mu - 1 = -n^2$, which is the value we obtained in (2.11) by introducing the quasi-symmetry approximation in the dynamic boundary condition. We can now test the approximation quantitatively by comparing the correct value of the frequency in (4.6) with the approximate value for any given value of s . Atmospheric inversions seldom exceed a temperature jump of 10°C . The corresponding s -value is less than 0.02, so practically no error is made by putting $1-s^2 = 1$ in (4.6). The quasi-symmetric value of the frequency differs from the correct value by an amount roughly equal to s . The quasi-symmetric theory is good to this approximation.

When the density difference between the layers is appreciable (as for example the interface between the ocean and the atmosphere), the dynamic

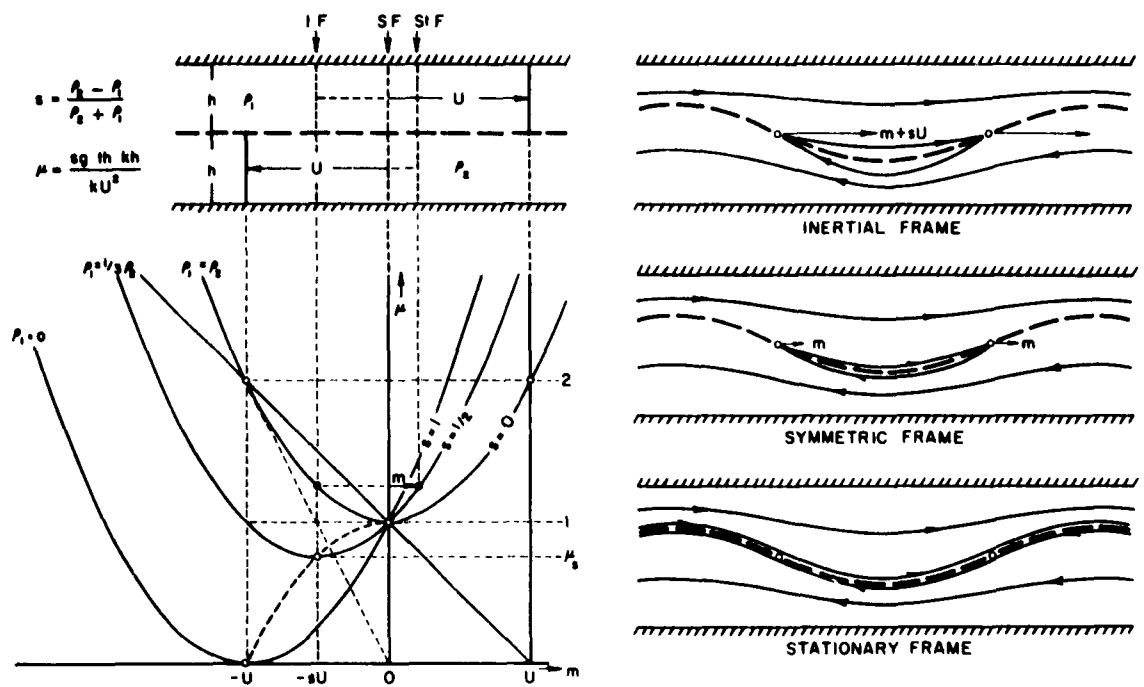


FIG. 4 DISPERSION DIAGRAM FOR NEUTRAL MODES IN A VORTEX SHEET

asymmetry across the interface can of course not be ignored. In the limiting case with zero density in the upper layer ($s = 1$) the system becomes one single layer with a free surface, and the frequencies of the normal modes in this layer are given by

$$(m-1)^2 = \mu = (g/kU^2)thkh.$$

- The normal modes (4.3) are here two families of neutral gravity waves which move in "the symmetric frame" toward negative x with the non-dimensional phase velocities $m = 1 \pm \sqrt{\mu}$. The fluid in the layer moves in the symmetric frame toward negative x with the speed $U = 1$. Relative to the fluid the modes move of course symmetrically in opposite directions with the same speed, $U^2 = (g/k)thkh$.

When the densities in the layers have arbitrary values with the heavier fluid in the lower layer, the static stability s has an intermediate value between the limiting values zero and one, and the normal modes of the system have the frequencies in (4.6). The long modes, $\mu > 1 - s^2$, have real frequencies, whose values may be represented graphically in a non-dimensional m, μ -diagram (see fig. 4) by a parabola whose vertex has the coordinates

$$\begin{aligned} m_s &= s, \\ \mu_s &= 1 - s^2. \end{aligned} \quad (\text{vertex of dispersion parabola})$$

The parabola has the same shape for all values of s . As s is changed continuously from zero to one, the vertex moves along a similar inverted parabola, $m_s^2 = 1 - \mu_s$, which connects the vertex points of the limiting vertex parabolas ($s = 0, 1$). With the coordinates of the vertex substituted, the frequency equation (4.6) becomes

$$(4.7) \quad (m - m_s)^2 = \mu - \mu_s.$$

The long ($\mu > \mu_s$)-modes in (4.3) are neutral waves which propagate in the symmetric frame with the non-dimensional phase velocities m in (4.7). Let

us now use another frame of reference which moves toward negative x relative to the symmetric frame with the speed $U_s = m_s U = sU$. In this frame the non-dimensional phase velocity of the mode is $m_0 = m - m_s$, and the dispersion equation for the mode has the simplest possible form, namely

$$(4.8) \quad m_0^2 = \mu - \mu_s.$$

The frame moves relative to the symmetric frame with the speed U_s such that $(\rho_0 + \rho_1)U_s = \rho_0 U - \rho_1 U$. It is the frame in which the center of mass of the system is stationary. The long stable neutral modes ($\mu > \mu_s$) move in opposite directions relative to the center of mass of the system with the same phase velocity $(\mu - \mu_s)^{1/2}$. The short unstable ($\mu < \mu_s$)-modes have stationary phase relative to the center of mass and they grow or decay at the exponential rate $(\mu_s - \mu)^{1/2}$. The transitional ($\mu = \mu_s$)-mode is a neutral wave which is stationary relative to the center of mass.

All the dispersion (or frequency) parabolas in fig. 4 have one point in common, with the coordinates $m = 0$, $\mu = 1$. This means that, regardless of the density difference between the layers, one of the stable modes of the system is stationary in the symmetric frame, namely the mode for which $U^2 = (sg/k) \tanh kh$. If for this stationary mode the direction of the flow is reversed in one of the layers, the acceleration remains unchanged everywhere. So the new situation is also stationary. But this is the field of a gravity wave in the frame which moves with the speed U of the wave relative to the fluid in the layers. In other words, the gravity wave propagates through the fluid with the speed U which makes its Richardson number μ in (4.5) equal to unity. This speed gives the overweight the right time to reverse the vorticity of the interface filaments during their passage from crest to trough.

If the wave is very much shorter than the depth of the layers ($kh \gg 1$), the hyperbolic factors in (4.1 and 4.2) are very nearly equal to the simple exponential functions, $\exp(\mp kz)$, and the Richardson number of the wave in

(4.5) has very nearly the value sg/kU^2 of the unbounded system. The boundaries have no appreciable effect on the waves which are much shorter than the depth of the layers. The short wave approximation with the boundary effects ignored is in fact quite good if the depth of the layer is more than half a wave length. For waves which are very much longer than the depth of the layers ($kh \ll 1$) we obtain another long-wave approximation by ignoring squares and higher powers of kh in the general formulas. It turns out that the same long-wave approximation is obtained when we ignore the vertical acceleration in the vertical dynamic equation, and take the pressure distribution along the vertical to be hydrostatic.

5. The quasi-static approximation. - Let us first consider the approximation as a long wave approximation of the general theory in section 4. If $kh \ll 1$ we may develop the hyperbolic functions (4.1 and 4.2) in series and ignore higher powers of kh . The streamfunction in the upper layer to the order of $(kh)^2$ is

$$(5.1) \quad \psi = \psi_1 \left(1 - \frac{z}{h}\right) \left[1 + \frac{1}{6} \frac{z}{h} \left(\frac{z}{h} - 2\right) (kh)^2\right] \quad (z > 0).$$

To the same order of approximation the upper horizontal velocity field is

$$(5.2) \quad hu = \psi_1 \left[1 + \left(\frac{1}{2} \frac{z^2}{h^2} - \frac{z}{h} + \frac{1}{6}\right) (kh)^2\right] \quad (z > 0).$$

The Richardson number of the wave, to the same approximation, is

$$(5.3) \quad \mu = (sgh/U^2) [1 - \frac{1}{6} (kh)^2].$$

If the wave is much longer than h , the factors in the brackets in these formulas differ from unity by an amount less than $(kh)^2$. The long wave approximation is obtained by taking these factors equal to one. The error in this approximation is only a fraction of $(kh)^2$.

We note that the long-wave approximation gives a motion with no vertical shear. Since the motion is irrotational it must have enough shear to compensate

for the curvature of the streamlines. But the streamlines have very slight curvature in the very long waves, and the shear is therefore so slight in the long waves that it may be ignored without significant error.

The long wave approximation gives the vertical velocity at an arbitrary level in the upper layer as

$$(5.4) \quad w = \left(1 - \frac{z}{h}\right) \frac{\partial \psi}{\partial x} = (h - z) \frac{\partial u_1}{\partial x} . \quad (z > 0)$$

We note that this approximation satisfies the solenoidal condition $\nabla \cdot \underline{v} = 0$ for the incompressible fluid. On the other hand, the lamellar condition $\nabla \times \underline{v} = 0$ is slightly violated, since the wind shear of the slightly curved flow is ignored. The vertical velocity in (5.4) increases linearly from zero at the upper rigid boundary to its maximum value at the interface level, namely

$$(5.5) \quad w_1 = h \frac{\partial u_1}{\partial x} = - \frac{Dh_1}{Dt} = \frac{Dz_B}{Dt} .$$

In the absence of shear the vertical fluid column remains vertical, and its height h_1 decreases at the rate of the vertical motion at its base, which in turn is equal to the horizontal divergence in the column.

Let us now compare this long wave approximation with the quasi-static approximation which is made when the dynamic effect of the vertical acceleration on the pressure is ignored, so the pressure distribution along the vertical is taken to be hydrostatic. The horizontal and vertical dynamic equations for quasi-static flow are therefore

$$(5.6) \quad \begin{aligned} \rho D\underline{v}/Dt &= - \nabla p, \\ \rho g &= - \partial p / \partial z. \end{aligned} \quad (\text{Quasi-static equations})$$

Let us apply these equations to an arbitrary motion of a homogenous fluid layer. Since the pressure distribution along the vertical is hydrostatic, the vertical thickness of the isobaric layers is the same everywhere in a homogenous layer. All the isobaric surfaces have the same shape and may be generated by displacing one of them along the vertical. This means that the

horizontal pressure gradient has no variation with height, and therefore the horizontal acceleration has no variation with height in a quasi-static motion of a homogenous fluid layer. If the motion has no vertical shear at a certain time, no shear can develop later. The vertical fluid columns will remain vertical at all times. We have seen that this condition is very nearly satisfied in waves which are much longer than the depth of the layer, so the motion in these waves obeys the quasi-static approximation with good accuracy. The kinematic conditions in (5.5) which were derived from the long wave approximation may also be derived from the quasi-static theory, because they are the consequence of the condition that the motion has no vertical shear. In other words, the long wave approximation is equivalent to the quasi-static approximation, and vice versa. It may be noted from (5.5) that the ratio between the amplitudes of the vertical and the horizontal components of the motion is equal to kh , that is the ratio between the vertical and horizontal scale of the motion. Quite generally then, the quasi-static equations (5.6) may be used as a good approximation for the description of the motion if the ratio between the horizontal and vertical scale of the motion is small. The percentual error we make by using these equations has the order of the square of this ratio. We shall show later in Chapter II that the same condition for the quasi-static approximation applies to compressible flow with a continuous variation of the density along the vertical.

The Richardson number has the constant value sgH/U^2 for all waves which are long enough to obey the quasi-static approximation with good accuracy. This means that the long quasi-static gravity waves in the resting double layer all move with the same phase speed C_L through the fluid, where $C_L^2 = sgH$. The quasi-static waves are non-dispersive. This speed exceeds the correct value by about $0.7(kh)^2$ per cent. The phase speed C_L of the long quasi-static

gravity waves may be taken as a convenient measure for the static stability of the resting double layer. When the layers have the relative translation $2U$, the Richardson number for the corresponding quasi-static Helmholtz waves is

$$(5.6) \quad \mu = sgh/U^2 = (C_L/U)^2. \quad (\text{Quasi-static})$$

It is a non-dimensional constant which includes all the physical parameters of the system, and it measures the ratio between the static stability and the shear across the interface. In the quasi-symmetric systems ($s \ll 1$), the Richardson number defines the dynamic properties of the quasi-static waves: If the static stability dominates ($\mu > 1$) the quasi-static waves are stable and have the frequency $m = \sqrt{\mu - 1}$. If the shear dominates ($\mu < 1$) the quasi-static waves are unstable with the growth rate $n = \sqrt{1 - \mu}$. In the systems with arbitrary densities in the layers the transition from stable to unstable quasi-static waves occurs at a Richardson number less than unity, namely $\mu_s = 1 - s^2$.

6. Lateral diffusion of vorticity from a shear layer. - In order to see how the vorticity diffusion effects the dynamic behavior of a vortex sheet, we shall examine how the undisturbed vortex sheet is modified by the diffusion process.

Let us begin by considering a more general system with the vorticity concentrated in a central layer between unbounded irrotational layers. The boundaries between the central layer and the outer layers are not sharp. The vorticity is supposed to decrease symmetrically in both directions from a maximum value at the center of the shear layer and approach zero asymptotically with increasing distance from the center level (see fig. 6).

The diffusion process is particularly simple if the vorticity distribution at some initial moment is given by the Gaussian error function

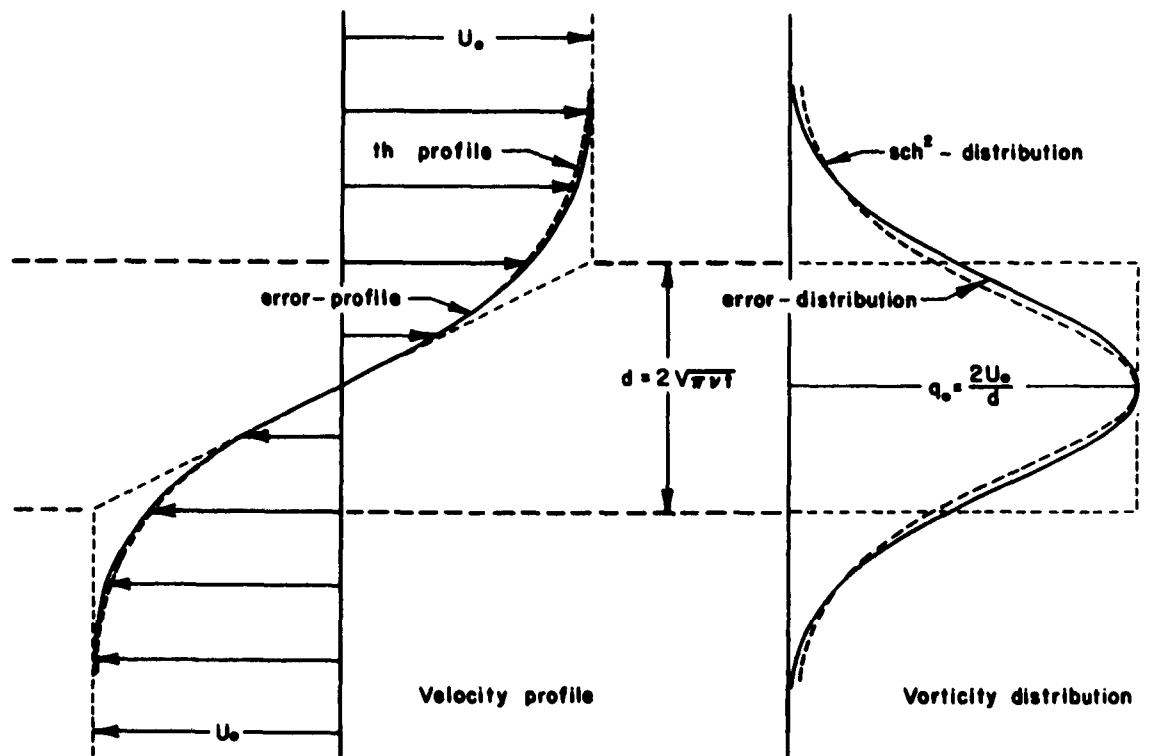


Fig. 6 Symmetric vorticity distribution across a central layer

$$(6.1) \quad q = q_0 \exp[-\pi(z/d)^2] = dU/dz.$$

In the symmetric frame which moves with the fluid at the central level the asymptotic maximum speed in the outer layers is

$$U_0 = \int_0^{\infty} q \, dz = q_0 \int_0^{\infty} e^{-\pi(z/d)^2} dz = \frac{1}{2} q_0 d.$$

With this value of q_0 substituted, the Gaussian vorticity distribution in (6.1) becomes

$$(6.2) \quad q = (2U_0/d) \exp[-\pi(z/d)^2].$$

The parameter d may be regarded as the depth of the central layer in an "equivalent" three layer system with the constant central level vorticity in a shear layer of depth d between unbounded irrotational outer layers which have the relative translation $2U_0$.

If the vorticity distribution in the layer is Gaussian at any one time, it will remain Gaussian at all times, while the depth grows with time. This is readily seen by substitution of q from (6.2) into the Navier-Stokes vorticity equation for two-dimensional shear flow, namely

$$(6.3) \quad \dot{q} = \nu \nabla^2 q = \nu q'' \quad (\nu = \text{kinematic viscosity})$$

Here the accent denotes height differentiation and the dot denotes local time differentiation in the symmetric frame. The logarithmic form of (6.2) is

$$\ln q = \ln(2U_0) - \ln d - \pi(z/d)^2.$$

The logarithmic height derivative is

$$q'/q = -2\pi z/d^2.$$

The second height derivative is

$$\frac{q''}{q} = \left(-\frac{1}{d} + \frac{2\pi z^2}{d^3} \right) \frac{2\pi}{d}.$$

The logarithmic time derivative is

$$\frac{\dot{q}}{q} = \left(-\frac{1}{d} + \frac{2\pi z^2}{d^3} \right) \dot{d}.$$

With these substituted the vorticity equation (6.3) gives

$$(6.4) \quad \dot{d} = 2\pi\nu/d,$$

which has the integral

$$d^2 - d_0^2 = 4\pi\nu(t - t_0).$$

The rate of growth of the depth of the Gaussian shear layer is proportional to the inverse depth of the layer. We may choose the initial shear layer as thin as we like. In the limit, if we choose $d_0 \rightarrow 0$, we have initially a vortex sheet, and we see that the vortex sheet immediately is changed by the viscous vorticity diffusion into a shear layer of finite depth, and this layer grows in thickness at the rate in (6.4). It is clear then that the dynamic theory for the instability of the vortex sheet in section 1 (which is based on the assumption that the vortex sheet retains its identity at all times) is not very satisfactory. To improve on the theory we must examine the instability of a shear layer of finite depth.

7. The instability of a constant shear layer. - The Gaussian shear layer which develops by the vorticity diffusion from a vortex sheet is not easy to discuss mathematically. We shall therefore in this section examine a shear layer of depth d with the constant wind shear,

$$q = 2U/d,$$

between unbounded irrotational outer layers which have the relative translation $2U$. All three layers have the same constant density. We shall ignore the vorticity diffusion at the boundaries of the shear layer.

Let the upper interface at some initial time be given a small amplitude sinusoidal deformation,

$$(7.1) \quad z_1 = A_g \cos kx, \quad (kA_g \ll 1)$$

while for the moment the lower interface is undeformed. Since the vorticity

is conserved in this system (see 6.3), this deformation means that the constant vorticity in the shear layer has been added in the sinusoidal segments above the interface level, and the same vorticity is removed from the segments below the interface level.

The field associated with the sinusoidal deformation of the upper interface may be represented as the resultant of two component fields. One component is the undisturbed shear layer with the undeformed interface. The second component, the γ -field, is irrotational everywhere except in the areas of added vorticities between the interface level and the deformed interface. It is easy to see what the general appearance of this periodic field must be. It has a slight asymmetry with reference to the interface level since all the positive vorticity is above and all the negative vorticity is below the interface level. However, if the interface amplitude is a small fraction of the wave length, this asymmetry may be ignored (the linear approximation). The field is then very nearly symmetric with reference to the interface level with sinusoidal distribution of vortex filaments along the interface level and zero vorticity elsewhere.

All the vortex filaments may be grouped in pairs distributed symmetrically with reference to any one of the nodes. The vertical line through the node is the central straight streamline of each pair, so it is a streamline of the resultant periodic γ -field.

It is readily seen that this periodic field will propagate the deformation upwind relative to the fluid at the interface. The wave speed is obtained by application of Stokes's theorem to a small rectangular section of the sinusoidal segment of added vorticity. The area of this section is $z_1 dx$. The contribution to the circulation round the segment along the vertical sides, $z_1 w$, are small of the second order. The circulation round the

area $z_1 dx$ to the first order (linear) approximation is

$$(7.2) \quad (u^+ - u^-)dx = qz_1 dx = (2U/d)z_1 dx. \quad (\text{The dynamic condition})$$

Here u^+ denotes the tangential component of the Laplacean ψ -field above the sinusoidal segment of added vorticity and u^- denotes the tangential component below. This is the dynamic condition for the deformed interface.

Let the ψ -field be represented by a streamfunction (as in 1.2), whose value at the interface level ($z=0$) be denoted by ψ_1 . Above and below the interface the streamfunction (as in 1.3) has the values

$$\psi^+ = \psi_1 e^{-kz}, \quad \psi^- = \psi_1 e^{kz}.$$

The tangential wind shear across the segment of added vorticity is then

$$u^+ - u^- = 2k\psi_1.$$

With this value substituted the dynamic condition (7.2) takes the simple form

$$(7.3) \quad \psi_1 = (U/k)z_1 \quad (k = kd)$$

This field propagates the deformation upwind relative to the fluid with a certain phase velocity C_0 . In a frame of reference which moves with this wave the wave is stationary and the deformed interface is one of the streamlines, having the slope

$$\frac{\partial z_1}{\partial x} = \frac{w_1}{C_0} = \frac{1}{C_0} \frac{\partial \psi_1}{\partial x}, \quad \text{or} \quad \psi_1 = C_0 z_1. \quad (\text{The kinematic condition})$$

This is the kinematic condition at the interface. The kinematic and dynamic conditions combined give the intrinsic upwind phase velocity of the wave at the moment when only the upper interface is deformed, namely

$$(7.4) \quad C_0 = U/k, \quad C_0/U = L/2\pi d.$$

This wave is called a Rayleigh wave after Lord Rayleigh who discovered it in 1880.*

The ψ -field associated with the added vorticity at the upper deformed interface gives a vertical motion to the fluid particles at the lower initially

* Lord Rayleigh: On the Stability and Instability of Certain Fluid Motions. Scientific Papers. Article 66, Vol. I, p. 474.

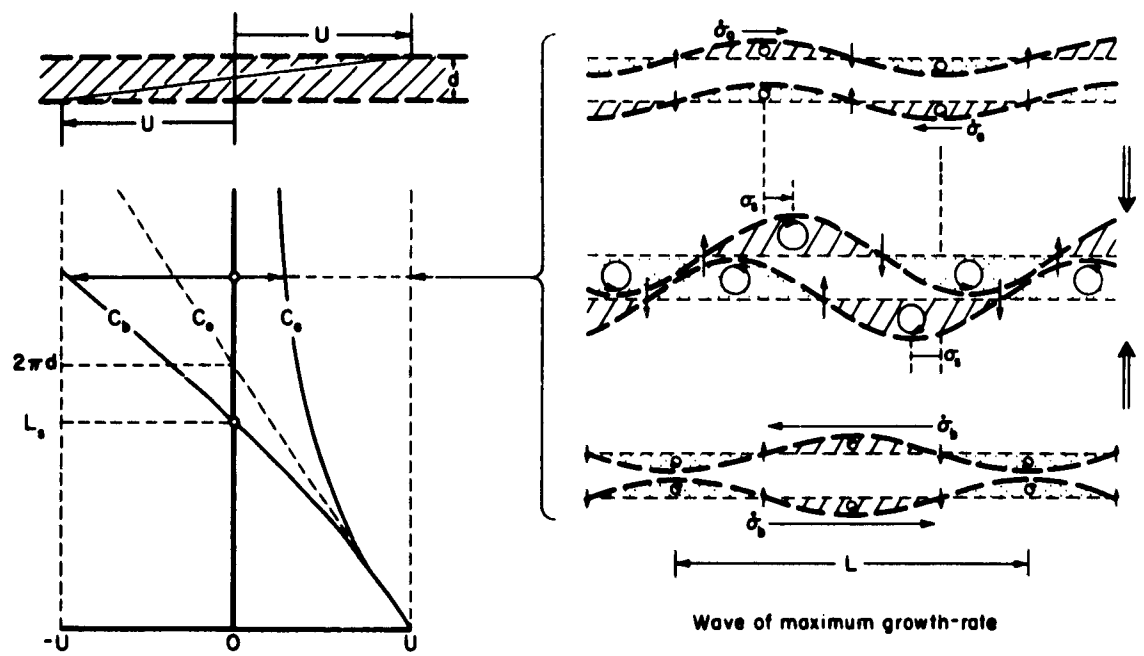


Fig. 7 Evolution of the long unstable Raleigh wave

level interface. In the next instant the lower interface will therefore have a sinusoidal deformation with crests and troughs in phase with the upper interface nodes. This lower deformation in turn is associated with a periodic component field of the type (7.3) centered at the lower interface. The evolution of the wave is therefore governed by the resultant of the two periodic fields from the added vorticities at both interfaces, and this evolution is not simple.

However, if at the initial moment both interfaces are given equal sinusoidal deformations with the same amplitude and phase we have a symmetric wave. Because of the kinematic and dynamic symmetry of the shear layer this wave will remain symmetric at all times. Similar to the Helmholtz wave in the vortex sheet (see fig. 1) we call the non-tilting state of the symmetric Rayleigh wave with the interfaces deformed in phase and the upper and lower component fields in opposite phase the a-state of the wave. The second non-tilting state with the interfaces deformed in opposite phase and the component field in phase is called the b-state. (See the wave diagrams in fig. 7.)

In the a-state the upper and lower component fields have opposite phase. The straight vertical streamlines of each component (passing through the interface nodes) coincide but they have opposite sense. The upper component, if operating alone, would propagate the upper deformation intrinsically upwind through the fluid with the speed U/κ . The lower component field tends to oppose this propagation. However, since the intensity of the field decreases exponentially with the distance from the interface, it is reduced by the factor,

$$(7.5) \quad \alpha = e^{-\kappa}, \quad (\kappa = kd)$$

at the upper interface level. Its contribution to the propagation of the upper wave is reduced by the same factor. The upper deformation therefore propagates

intrinsically upwind through the a-state with the speed

$$(7.6) \quad C_a = U(1 - \alpha)/\kappa.$$

But the fluid at the interface moves downwind with the speed U in the symmetric frame, so the upper wave moves downwind through the a-state with the speed $U - C_a$ in the symmetric frame. Because of the symmetry the lower wave has the same downwind speed.

In the b-state with the interfaces deformed in opposite phase (see the lower wave diagram in fig. 7) the upper and lower component fields have the same phase. They propagate the deformations intrinsically upwind through the fluid with the speed

$$(7.7) \quad C_b = U(1 + \alpha)/\kappa.$$

In the symmetric frame the symmetric wave moves downwind through the b-state with the speed $U - C_b$. The arithmetic mean of the intrinsic phase velocities in the non-tilting states is

$$(7.8) \quad \frac{1}{2}(C_a + C_b) = U/\kappa = C_0.$$

It is readily seen that the wave has this intrinsic speed in a state half way between the non-tilting states.

The phase velocities in the non-tilting states are represented graphically by the C_a -line and the C_b -line in the diagram to the left in fig. 7. The C_0 -line is the straight broken line through the origin and the point $(2\pi d, U)$. It represents the contribution to the propagation of the deformation from its own local field. This contribution is proportional to the wave length. The C_a -line and the C_b -line are located symmetrically on either side of this line. For waves shorter than the double depth of the shear layer ($\kappa > 3$) the exponential term $\alpha = e^{-\kappa} < 0.04$, so for these short waves the C_a - and C_b -lines are practically coincident with the central C_0 -line. The reason is that the intensities of the component fields are insignificant at the level

of the other interface in the very short wave, so each component wave behaves as if the other were absent. As the wave length is increased the action from the other component field becomes increasingly important. The C_a -line departs more and more from the central C_0 -line, and the C_b -line departs the same distance to the other side. For the $(\kappa=1)$ -wave the departure is U/e . Ultimately for waves which are very much longer than the depth of the shear layer $(\kappa \ll 1)$ we may develop the exponential term in series and ignore the higher terms

$$C_a = \frac{U}{\kappa}(1 - e^{-\kappa}) \rightarrow U(1 - \frac{1}{2}\kappa). \quad (\text{for } L \gg 2\pi d).$$

The C_a -line has the long wave asymptote $C_a = U$. Using this result in (7.8) we find that the C_b -line has the asymptote $C_b \rightarrow U(2/\kappa - 1)$. It meets the C_a -asymptote at $\kappa=1$ and has twice the slope of the C_0 -line.

Since $C_a < U$ for all wave lengths, all waves move downwind through the a-state in the symmetric frame. As soon as the wave leaves the a-state the lower field begins to augment the amplitude of the upper deformation and vice versa. The wave starts growing when it leaves the a-state. In the a-state the shear layer as a whole has a sinusoidal deformation with uniform depth everywhere. As the wave moves downwind from the a-state the shear layer becomes periodically inflated and deflated with the inflated parts centered downwind from the interface crests. In these respects the shear layer and the vortex sheet have a similar behavior. In both the symmetric wave moves downwind through the a-state while the vorticity becomes concentrated towards the nodes downwind from the crests.

The wave of $C_b = U$ has a stationary neutral b-state. This wave has the wave number

$$(7.9) \quad C_b = U: \quad \kappa_s = 1 + e^{-\kappa_s} = 1.2785 \dots \quad (L_s = 4.9d).$$

It is the wave length for which the C_b -line meets the U -line in fig. 7.

Waves shorter than the stationary neutral b-wave move downwind through both non-tilting states, so they move progressively downwind at all times. They pass through the a-state with maximum speed and minimum amplitude. They grow and slow down after leaving the a-state and pass through the b-state with minimum speed and maximum amplitude. They are stable waves with rhythmic variations of speed and energy between the extreme values in the non-tilting states.

The longer ($\kappa < \kappa_s$)-waves move downwind through the a-state and upwind through the b-state. From either non-tilting state they approach asymptotically a state of stationary phase with a downwind tilt from the a-state, as shown in fig. 7 for the wave $\kappa = 0.8$. As this asymptotic state of stationary phase is approached the wave keeps on growing, ultimately at the rate which is characteristic for the state of stationary phase. This growth rate must be proportional to the instantaneous strength of the component fields, and hence to the instantaneous interface amplitude A_s . Denoting the proportionality factor by n , the instantaneous growth rate is $dA_s/dt = nA_s$. In the asymptotic state of stationary phase the long symmetric waves in the shear layer grow at a constant exponential rate.

The evolution of the symmetric wave from an arbitrary state is obtained by applying the boundary conditions to such a state. The deformations of the interfaces in an arbitrary symmetric state are

$$(7.10) \quad \begin{aligned} z_1 &= A_s \cos(kx - \sigma) = \text{real part of } (\xi e^{ikx}), \\ z_2 &= A_s \cos(kx + \sigma) = \text{real part of } (\xi^* e^{ikx}). \end{aligned}$$

The phase σ is measured positive downstream from the a-state. The complex wave parameter ζ has the value

$$(7.11) \quad \zeta = A_s e^{-i\sigma} = \xi - i\eta.$$

The deformation z_1 of the upper interface is associated with a Laplacean

field whose streamfunction has the interface level value $\psi_1 = (U/\kappa)z_1$, and whose intensity at the lower interface is reduced by the factor $\alpha = e^{-\kappa}$. The kinematic conditions are that the interface particles are moved jointly by the two component Laplacean fields associated with the interface deformations. The vertical velocity at the upper interface in non-dimensional units ($k = U = 1$) is

$$(7.12) \quad w_1 = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) z_1 = \frac{\partial}{\partial x} \left(\frac{z_1}{\kappa} - \alpha \frac{z_2}{\kappa} \right).$$

With the deformations of the symmetric wave in (7.10) substituted, its amplitude becomes

$$\dot{\zeta} + i\zeta = i(\zeta - \alpha\zeta^*)/\kappa.$$

The real and imaginary parts of this equation are the evolution equations for the symmetric wave. They are

$$(7.12) \quad \begin{aligned} \dot{\eta} &= (1 - n_a)\xi, & n_a &= (1 - \alpha)/\kappa \\ \dot{\xi} &= -(1 - n_b)\eta, & n_b &= (1 + \alpha)/\kappa \end{aligned}$$

The parameters n_a and n_b are the non-dimensional intrinsic upstream phase velocities in the non-tilting states of the wave. These equations are equivalent to the corresponding equations for the symmetric Helmholtz waves in (2.7). They have the same form when the values of ξ, η are substituted from (7.11), namely

$$(7.13) \quad \begin{aligned} \frac{d}{dt} \ln A_s &= (1 - n_a - \delta) \cot \sigma, \\ \frac{d}{dt} \ln A_s &= (n_b - 1 + \delta) \tan \sigma. \end{aligned}$$

They may be integrated in precisely the same manner, to find the evolution from an arbitrary state of the wave. In particular, if the growth rate in the states of stationary phase ($\delta = 0$) is denoted by n we get

$$(7.14) \quad \begin{aligned} n \tan \sigma_s &= 1 - n_a = \delta_a, \\ -n \cot \sigma_s &= 1 - n_b = \delta_b, \end{aligned}$$

precisely as in (3.1) for the Helmholtz waves. With the values of n_a and n_b

substituted from (7.12) the growth rate n , with the proper dimensions kU restored, becomes

$$(7.15) \quad (n\kappa/kU)^2 = (nd/U)^2 = \alpha^2 - (1 - \kappa)^2 = e^{-2\kappa} - (1 - \kappa)^2.$$

It is zero for $\kappa = 1 \pm e^{-\kappa}$, that is for the stationary b-wave κ_s in (7.9) and for an infinitely long wave ($\kappa \rightarrow 0$). Between these limits n in (7.15) is real, and the waves are unstable. The mode of maximum growth rate ($dn/d\kappa = 0$) has the wave number

$$(7.16) \quad \kappa_m = 1 - \exp(-2\kappa_m) = 0.8. \quad (L_m = 7.9d)$$

It is about eight times longer than the depth of the shear layer. From (7.15) its growth rate is

$$(7.17) \quad n_m = 0.4 U/d = \frac{1}{2} U\kappa_m. \quad (L_m = 7.9d)$$

It is very nearly half the growth rate of the same wave if the depth of the shear layer shrinks to zero and becomes a vortex sheet. The very long modes (to the order of κ) have the growth rate $n = kU$, the same as in a vortex sheet.

From (7.14) the stationary phase is given by

$$\tan^2 \sigma_s = \frac{\delta_a}{\delta_b} = \frac{\alpha - (1 - \kappa)}{\alpha + (1 - \kappa)}; \quad \text{or} \quad \cos 2\sigma_s = (1 - \kappa)e^{\kappa}.$$

The κ_s -wave has a stationary b-state ($\sigma_s = 90^\circ$). The growing ($\kappa = 1$)-mode is 45° downstream from the a-state. The mode of maximum growth rate ($\kappa_m = 0.8$) has the phase of $\cos 2\sigma_m = \sqrt{0.2}$. It is $31^\circ 45'$ downstream from the a-state. This wave is shown in fig. 7.

Waves which are very much longer than the depth of the shear layer move very slowly downstream through the a-state and very fast upstream through the b-state, so their stationary phase is near the a-state. To the order of κ the stationary phase of the very long waves is $\sigma_s = \frac{1}{2}\kappa$. The upper crest is displaced the distance d downstream from the lower crest, and the growth rate (to the same order) is kU . The resultant streamfunction at the upper interface level, with higher powers of κ ignored, is

$$\psi_1 = A_g [\cos x - \alpha \cos(x + \kappa)] / \kappa = \sqrt{2} A_g \cos(x - 45^\circ).$$

The streamline crest is 45° downstream from the interface crest, the same as in the growing mode of the vortex sheet. In the limit as $d \rightarrow 0$ the value of the streamfunction and the growth rate kU become exact for all wave lengths. The shear layer behaves approximately as a vortex sheet for waves which are very much longer than the depth of the layer.

Let us now consider the justification of the linear approximation which has been used in the above theory. It is based on the assumption that the wave amplitude is much smaller than the wave length ($kA_g \ll 1$). In the short stable waves the amplitude cannot grow beyond the b-state amplitude. If $kA_{sb} \ll 1$ the linear approximation is good during the entire oscillation of the short stable wave, and the linear theory is valid for these oscillations. On the other hand, a long unstable wave should, according to the linear theory, from any state except the decaying state of stationary phase (the decaying mode), move toward the growing state of stationary phase (the growing mode) where ultimately it grows at an exponential rate. After a certain time the condition, $kA_g \ll 1$, is no longer satisfied, and the linear theory is no longer valid. The subsequent non-linear evolution of the unstable waves is a difficult mathematical problem, but its main characteristics may be anticipated by qualitative physical reasoning.

Let us consider the wave of maximum growth rate shown in fig. 7. As this wave leaves the a-state, the shear layer becomes periodically inflated and deflated with the inflated parts centered downwind from the interface crests. As the wave approaches the state of stationary phase, and the interface keeps on growing at an exponential rate (according to the linear theory), the kinematics of the wave place an upper limit on the linear evolution, namely the moment when the deflated parts shrink to zero in the middle. Let

the wave of maximum growth rate be started from an a-state with an amplitude which is one per cent of the wave length ($A_{sa} = 0.01 L_m$, or $k_m A_{sa} = 0.06$). The linear theory is then initially quite good. If the linear evolution is extrapolated up to the kinematic limit, the wave will have reached very nearly to the state of stationary phase at that time, so the kinematically limiting interface amplitude A_{sm} is given by the formula, $A_{sm} \sin \sigma_m = \frac{1}{2}d$, and we have

$$(7.18) \quad A_{sm} k_m = \frac{1}{2} \kappa_m / \sin \sigma_m = 0.76. \quad (\kappa_m = 0.8, \sigma_m = 31^\circ 45')$$

This value does not satisfy the condition $A_g k \ll 1$ for the linear approximation. Near the kinematic limit the evolution of the wave is certainly no longer linear. The non-linear terms which have been ignored are already becoming increasingly important. To get a rough idea of the nature of this non-linear evolution we shall nevertheless extrapolate the linear evolution up to the time of the kinematic limit and then use a different kind of reasoning for the subsequent evolution.

Let us first determine the time which the wave needs to grow from the initial a-state of $A_{sa} = 0.01 L_m$ to the kinematically limiting amplitude $A_{sm} k_m = 0.76$, or $A_{sm}/A_{sa} = 12$. This time (see 3.3) is given by

$$\text{ch } n_m t = 12 \cos \sigma_m = 10.2 = \text{ch } 3, \quad \text{or} \quad n_m t = 3.$$

From (7.17)

$$n_m = \frac{1}{2} U k_m = \kappa / T_m, \quad (T_m = L_m / U).$$

Thus, from an initial a-state with the amplitude one per cent of the wave length, the wave of maximum growth rate reaches the kinematic limit in the state of stationary phase in the time $t = (3/\pi) T_m$, that is, roughly the time during which the outer fluid moves a wave length in the symmetric frame.

The subsequent non-linear evolution may be estimated as follows:

The entire field of motion may be regarded as the resultant of all the partial fields associated with the vorticity in the separated regions of rotating fluid. Let us first consider one of these partial fields alone and estimate how the fluid would move if all the other fields were absent. We represent this partial field as the resultant of two component fields, namely, one component field coming from the largest ellipse which can be inscribed in the separated rotational region, and the other component coming from the remaining vorticity within the two asymmetrically located arms which connect the ellipse with the points of separation. It is clear that the major axis of the ellipse is parallel to the interface tangents at the nodes, so from (7.18) its inclination is given by

$$\tan \theta = A_{sm} k_m = 0.76. \quad (\theta = 37^\circ 15').$$

The minor axis is one half of the normal distance between these tangents, that is

$b = \frac{1}{2} d \cos \theta + (\sigma_m / k_m) \sin \theta = \frac{1}{2} d \cos \theta (1 + 2\sigma_m A_{sm} k_m / \kappa_m) = 0.82 d$,
or $b = 0.82 d$. The analytical determination of the major axis is not quite so simple, but a careful graphical determination gives the value $a = 2.2 d$. The elliptic vortex covers the area $\pi ab = 5.7 d^2$. The entire separated rotational region includes the fluid within one wave length of the shear layer and hence covers the area $L_m d = 7.9 d^2$. The elliptic vortex has 72 per cent and the connecting arms 28 per cent of the vorticity of the partial field.

The field of an isolated elliptic vortex in an unbounded irrotational surrounding fluid was derived by Kirchhoff*. The elliptic boundary has a solid rotation in the sense of the vorticity inside with the angular speed

$$\Omega = q ab / (a + b)^2. \quad (\text{elliptic vortex})$$

* Kirchhoff, Mechanics XX; Lamb, Hydrodynamics, 1932, p. 232.

The inscribed elliptic vortex in the separated region of the shear layer would at the moment of separation rotate with the angular speed

$$\Omega = 0.2 q = 0.4 U/d = n_m$$

if all the other vorticity were absent.

Let us for the moment ignore the component field coming from the connecting arms, and consider how the partial fields from the other inflated separated regions would move the elliptic vortex. Each of these have the vortex strength $\kappa = 2 UL_m$, and their action in the region of the elliptic vortex is roughly the same as that of vortex filaments of the same strength placed at their respective centers. These filaments may be grouped symmetrically in pairs whose singular point coincides with the center of the elliptic vortex. Near this center all these partial fields are pure deformations whose principal axes are inclined 45° toward the central level. This resultant deformation field is stationary. At the moment of separation the major axis of the elliptic vortex is inclined 37° toward the level. A short time later it has turned into coincidence with the axis of inflow of the deformation field. The deformation field has therefore at first very little direct influence on the rotation of the ellipse. However, it deforms it into a more circular shape, and thereby causes its rotation to speed up. The deformation continues until the major axis of the ellipse is vertical and bisects the angle between the stationary axes of deformation. At that moment the ratio between the major and minor axis is a minimum and the ellipse has the fastest rotation from its own field. The deformation now also contributes to the rotation of the ellipse. This process can be examined more quantitatively:

The two members of the N^{th} pair of vortex filaments have the distance NL_m from the center of the ellipse. The field of this pair has the stream-

-function

$$\psi = (2U/k_m) \ln (R_1 R_2).$$

Using polar coordinates (R, θ) from the center of the ellipse, we have

$$(R_1 R_2)^2 = R^4 + (NL_m)^4 - 2(RNL_m)^2 \cos 2\theta.$$

In the region of the ellipse we may ignore higher powers of $(R/NL_m)^2$, and write

$$R_1 R_2 = (NL_m)^2 - R^2 \cos 2\theta.$$

Within this region the streamfunction has the ascendent

$$\tilde{\psi} = (4UR/k_m) [\sin 2\theta (R\tilde{r}) - \cos 2\theta \tilde{r}R] / (NL_m)^2.$$

The vector in the brackets is a unit vector so the speed of the field at the distance R from the center has the constant value

$$v = 2Rn_m (\pi N)^{-2} \quad (n_m = \frac{1}{2} U k_m).$$

This is the speed of deformation of the N^{th} pair. All the partial fields have the same orientation, so the resultant deformation has the speed

$$v = 2Rn_m \sum_1^{\infty} (\pi N)^{-2} = Rn_m/3^*$$

At the moment when the ellipse is oriented along the axes of deformation, its axes are changed at the instantaneous rates

$$\dot{b}/b = -\dot{a}/a = n_m/3,$$

and its rotation changes at the rate

$$\frac{\dot{\Omega}}{\Omega} = \frac{2}{3} \frac{a-b}{a+b} n_m.$$

Substituting here, for a rough estimate, the values at the time of separation, namely

$$a = 2.2 d, \quad b = 0.82 d, \quad \Omega_0 = n_m,$$

* H. B. Dwight: Tables of Integrals and Other Mathematical Data, Formula 48.2. MacMillan Co., N.Y., 1957.

the ellipse at that moment has the angular acceleration $\dot{\Omega} = 0.3 n_m^2$. Let us assume, again for a rough estimate, that the acceleration has this constant value while the ellipse turns from its inclined position at the moment of separation ($\theta = 37^\circ 15'$) to the vertical position ($\theta = 90^\circ$), and evaluate the corresponding time interval t . It is given by

$$0.15(n_m t)^2 + n_m t = (52.75/180)\pi = 0.92,$$

or $t = 0.82/n_m = 0.26 T_m$. If we ignore the acceleration and let the ellipse rotate with the constant initial angular speed n_m it would turn the same angle in the time interval $0.29 T_m$.

It remains to consider the component field from the vorticity in the two arms which connect the ellipse with the points of separation. We recall that these include about 28 per cent, that is, each arm about 14 per cent of the total vorticity in the separated inflated region. This component field is roughly the same as that of a pair of vortex filaments of the same strength placed at the centers of vorticity of the arms. At the moment of separation these filaments have roughly the distance $2.4d = 0.3 L_m$ from the center of the ellipse, and their center line is inclined about $22\frac{1}{2}^\circ$ toward the center level. In the region inside the ellipse the field of this pair is approximately a pure deformation with axes 45° inclined toward the center line. At the distance R from the center of the ellipse the speed of this deformation field is

$$v = 0.14 \cdot 2R n_m (0.3\pi)^{-2} \approx R n_m / 3.$$

This field has about the same strength in the region of the ellipse as the resultant deformation from all the other separated vorticity concentrations. The orientation of this field is such that it tends to slow down the rotation of the ellipse. Furthermore, whereas the deformation field from the other

vorticity concentrations is quasi-stationary, this latter field from the connecting arms follows the ellipse in its rotation. The two deformation fields effect the rotation of the ellipse in opposite sense, and since both effects are rather small, every elliptic vortex rotates roughly as if all the vorticity outside the ellipse were absent. It turns with the angular velocity n_m from the inclined position at the time of separation to the vertical position during the time the outer air moves about one quarter of a wave length. The ellipse is deformed toward a more rounded shape during this transition. We note that these conclusions resemble the results from Rosenhead's computations for the vortex sheet.

8. The statically stable shear layer. - We shall now examine a layer with constant shear between unbounded irrotational outer layers of different densities with the lighter fluid ($\rho = \rho_1$) above and the heavier fluid ($\rho = \rho_2$) below the shear layer. The density distribution in the shear layer may be assigned in different ways. In this section we consider the theoretically simplest model where the shear layer is homogenous with a density which is the arithmetic mean of the outer layer densities. In this model the boundaries of the shear layer have the same static stability

$$(8.1) \quad \frac{1}{2}s = \frac{1}{2}(\rho_2 - \rho_1)/(\rho_2 + \rho_1).$$

This system with arbitrary densities in the outer layers was investigated by G. I. Taylor* in 1916. The quasi-symmetric system ($s \ll 1$) was investigated later by Goldstein**. The quasi-symmetric shear layer is governed by a Richardson number which has the value

$$(8.2) \quad \mu = \frac{1}{2}sgd/U^2.$$

* Taylor, G. I., 1931: Effect of Variation in Density on the Stability of Superposed Streams of Fluids. Proc. Roy. Soc. A 132, pp. 499-523.

** Goldstein, S., 1931: On the Stability of Superposed Streams of Fluids of Different Densities. Proc. Roy. Soc. A 132, pp. 524-548.

We shall examine the behavior of symmetric waves in this quasi-symmetric shear layer. Similar to the statically neutral shear layer, the deformations of the boundaries generate Laplacean fields associated with the displaced vorticity in the shear layer. The interface level value of the component field associated with the upper deformation z_1 is $(U/\kappa)z_1$. However, the static stability across the interface now gives rise to sliding vorticity along the deformed interface whose change is governed by the dynamic condition in (2.4), namely,

$$\frac{D}{Dt}(u^+ - u^-) = sg \frac{\partial z_1}{\partial x}.$$

Let ψ_1 denote the interface level value of the streamfunction of the Laplacean field associated with this sliding vorticity. In terms of the streamfunction the dynamic condition has the form

$$(8.3) \quad \frac{D\psi_1}{Dt} = (\frac{1}{2}sg/k) \frac{\partial z_1}{\partial x}.$$

Let us find out under what conditions this sliding vorticity wave is stationary and neutral in the symmetric frame. The individual change is then equal to the convective change, so the dynamic condition becomes $\psi_1 = (\frac{1}{2}sg/kU)z_1$. The sliding vorticity wave is in phase with the interface deformation with maximum value at the crest. The field strength is such that the sliding vorticity of the interface filament is exactly reversed by the overweight while the filament moves from crest to trough in the symmetric frame. This is the wave mechanism of the neutral gravity wave, so we shall call this part of the symmetric wave a gravity wave. The second part, which comes from the displaced vorticity of the shear layer, has the field strength $(U/\kappa)z_1$ of the Rayleigh wave. We shall call this part of the symmetric wave a shear wave. The gravity wave is momentarily stationary and neutral in the symmetric frame if it is in phase with the shear wave with an amplitude which is μ

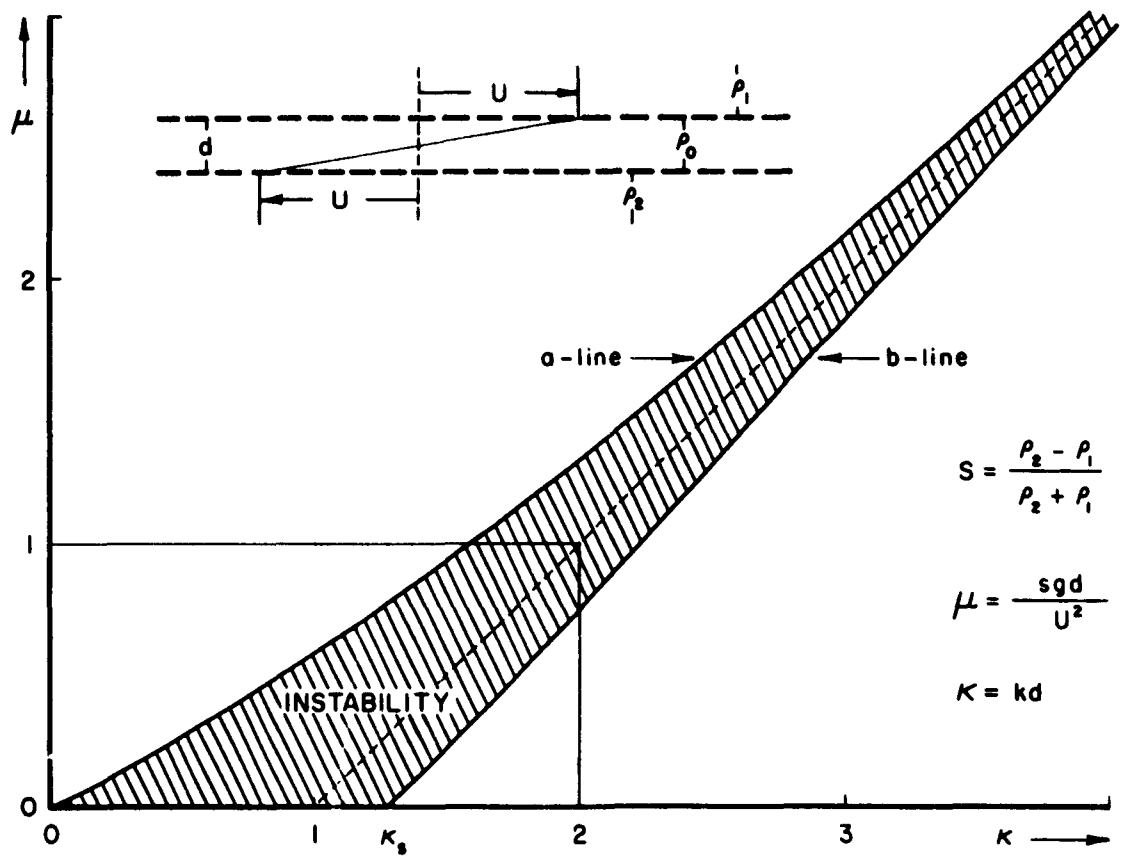


Fig.8 Stability diagram for a stratified shear layer (after Goldstein)

times the shear wave amplitude. This condition may be satisfied in an arbitrary tilting state of the symmetric wave. However, if the entire symmetric wave shall be stationary and neutral, this condition must be obeyed by both the gravity wave and the shear wave simultaneously, which means that the symmetric wave must be non-tilting. When the shear wave is stationary in a non-tilting state, the upper and lower fields will jointly move the interface deformations intrinsically upstream with the speed U , so the kinematic condition at the upper interface of the stationary wave is (see 7.12)

$$U \frac{\partial z_1}{\partial x} = (1 \pm \alpha) \left(\frac{U}{\kappa} + \frac{1}{2} \frac{sg}{kU} \right) \frac{\partial z_1}{\partial x}. \quad (\alpha = e^{-\kappa})$$

The statically stable shear layer accordingly has two stationary neutral waves, namely,

$$(8.4) \quad \begin{aligned} \text{Stationary a-wave:} \quad n_a (1 + \mu) &= 1, & n_a &= (1 - \alpha)/\kappa, \\ \text{Stationary b-wave:} \quad n_b (1 + \mu) &= 1. & n_b &= (1 + \alpha)/\kappa. \end{aligned}$$

The wave numbers of these stationary waves for any given Richardson number are shown by the two lines in the stability diagram in fig. 8. Both lines have the same asymptote, $\mu = \kappa - 1$. The a-line starts from the origin with the tangent slope $1/2$. Near the origin the stationary a-wave is very nearly given by

$$\mu = \frac{1}{2} \kappa, \quad \text{or} \quad sg/kU^2 = 1. \quad (\kappa = kd \ll 1)$$

If the depth of the shear layer shrinks to zero this formula is exact and gives the stationary neutral Helmholtz wave (2.2) which marks the transition between the long stable and the short unstable waves in the statically stable vortex sheet.

The b-line starts with unit slope from the stationary neutral Rayleigh wave which marks the transition between the long unstable and the short stable waves in the statically neutral shear layer. This makes it plausible

that the shear layer has unstable waves in the spectral interval between the two lines, as indicated in the figure.

9. The normal modes of the shear layer. - In an arbitrary state the symmetric wave has the wave elements

$$(9.1) \quad \begin{aligned} z_1 &= A_s \cos(kx - \sigma) = \text{Real part } (\zeta e^{ikx}), \\ \psi_1 &= (U/\kappa) A \cos(kx - \theta) = \text{Real part } (Z e^{ikx}). \end{aligned}$$

The complex wave parameters ζ and Z have the values

$$(9.2) \quad \begin{aligned} \zeta &= A_s e^{-i\sigma} = \xi - i\eta, \\ Z &= (U/\kappa) A e^{-i\theta} = X - iY. \end{aligned}$$

The amplitude A of the gravity wave is weighted by the factor (U/κ) . This makes A/A_s the amplitude ratio of the sliding vorticity to the shear vorticity, thus

$$(9.3) \quad r = \frac{A}{A_s} = \frac{\text{sliding vorticity amplitude}}{\text{shear vorticity amplitude}}.$$

The change of the gravity wave is governed locally by the overweight of the sloping interface filaments, so we may apply this condition in a local frame which moves with the filament. The individual change is the local change in this frame. In the units $U=k=1$ the dynamic condition (8.3) for the symmetric wave (9.1) is therefore $\dot{Z} = (\mu/\kappa)i\zeta$, which has the real and imaginary parts

$$(9.4) \quad \begin{aligned} \dot{X} &= (\mu/\kappa)\eta, & \text{or} & & \dot{A} \cos \theta - \dot{\theta} A \sin \theta &= \mu A_s \sin \sigma, \\ -\dot{Y} &= (\mu/\kappa)\xi, & \text{or} & & \dot{A} \sin \theta + \dot{\theta} A \cos \theta &= -\mu A_s \cos \sigma. \end{aligned}$$

Elimination of $\dot{\theta}$ gives the growth rate of the gravity wave, namely,

$$(9.5) \quad \dot{A}/A = (\mu/r) \sin(\sigma - \theta).$$

The gravity wave grows when it is upstream from the shear wave with the sliding vorticity maximum in the hill upwind from the interface crest.

Elimination of \dot{A} from (9.4) gives the intrinsic downwind phase velocity of the gravity wave relative to the fluid at the interface, namely,

$$\dot{\theta} = -(\mu/r) \cos(\sigma - \theta).$$

Since the fluid moves downstream with unit speed in the symmetric frame, the phase velocity in this frame is

$$(9.6) \quad \dot{\theta} = 1 - (\mu/r) \cos(\sigma - \theta).$$

If the wave has a state of stationary phase, $\dot{\theta} = \dot{\sigma} = 0$, the growth rate n and the amplitude ratio r_s in this state, obtained from the ratio and the sum of the squares of (9.5) and (9.6), are

$$(9.7) \quad \begin{aligned} n &= \tan(\sigma_s - \theta_s), \\ r_s^2 &= \mu^2/(1 + n^2). \end{aligned}$$

The behavior of the shear wave in an arbitrary state is not quite so simple because the boundaries of the shear layer are moved by all four Laplacean fields. However, if the shear wave has the same phase as the gravity wave ($\sigma = \theta$), it behaves at that moment as a Rayleigh wave in a statically neutral shear layer, only with the field intensities augmented by the factor $1 + r$. In particular, its phase velocity in a state of $\sigma = \theta$ (see 7.14) is

$$(9.8) \quad \dot{\sigma} = 1 - (1 + r)(n_a \cos^2 \sigma + n_b \sin^2 \sigma). \quad (\sigma = \theta)$$

In the non-tilting states both the gravity wave and the shear wave are instantaneously neutral, and they move through these states with the downwind phase velocities

$$(9.9) \quad \begin{aligned} \dot{\theta}_a &= 1 - \mu/r_a = n \tan \theta_s, & \dot{\sigma}_a &= 1 - n_a(1 + r_a) = n \tan \sigma_s, \\ \dot{\theta}_b &= 1 - \mu/r_b = -n \cot \theta_s, & \dot{\sigma}_b &= 1 - n_b(1 + r_b) = -n \cot \sigma_s. \end{aligned}$$

The first parts of these formulas apply to non-tilting states with arbitrary amplitude ratios. If the amplitude ratios have the values characteristic of a normal mode, the non-tilting phase speeds of the mode are related to the

growth rate and stationary phase of the mode by the formulas in (3.1), which have been repeated in (9.9). Both the gravity wave component and the shear wave component are stationary and have the same growth rate in the state of stationary phase of the mode (see 9.7), so the results derived in section 3 apply to both. In particular, the principle of constant energy transport of the component waves in (3.5),

$$\dot{\theta} A^2 = \dot{\theta}_a A_a^2; \quad \dot{\sigma} A_s^2 = \dot{\sigma}_a A_{sa}^2,$$

give the relation between amplitude ratio and phase velocities in an arbitrary state of the mode, namely,

$$(9.10) \quad (\dot{\theta}/\dot{\sigma}) r^2 = (\dot{\theta}_a/\dot{\sigma}_a) r_a^2 = (\dot{\theta}_b/\dot{\sigma}_b) r_b^2 = \pm r_a r_b.$$

or

$$(9.11) \quad \dot{\theta}_a/\dot{\sigma}_a = \dot{\sigma}_b/\dot{\theta}_b = \pm r_b/r_a.$$

These formulas apply to both unstable modes ($n^2 > 0$) and stable modes ($n^2 < 0$). In both, the phase velocity ratio is inverted from one non-tilting state to the other. This means that the stable mode has a tilting state where both component waves move with equal speeds ($\dot{\theta} = \pm \dot{\sigma}$). This state of the stable mode is the nearest equivalent to the state of stationary phase of the unstable mode (zero speed of both waves). We shall denote it by the subscript m. The amplitude ratio in the state of equal wave speeds (from 9.10) is the geometric mean of the ratios in the non-tilting states $r_m^2 = r_a r_b$. The phase of this state is obtained from the phase equations (3.4) for the component waves, namely,

$$\tan \theta = \tan \theta_s \tanh t, \quad \tan \sigma = \tan \sigma_s \tanh t.$$

Their ratio gives

$$(9.12) \quad \tan \theta / \tan \sigma = \dot{\theta}_a / \dot{\sigma}_a = \dot{\sigma}_b / \dot{\theta}_b$$

The tangents of the phase angles of the component waves have the same ratio in all states of the normal mode. The time derivatives of the phase equations in a state of equal speeds have the ratio

$$(9.13) \quad \cos^2 \theta_m / \cos^2 \sigma_m = \pm \dot{\sigma}_a / \dot{\theta}_a. \quad (\dot{\theta}_m = \pm \dot{\sigma}_m)$$

The product of these equations gives

$$\sin 2\theta_m \mp \sin 2\sigma_m = 2 \sin (\theta_m \mp \sigma_m) \cos (\theta_m \pm \sigma_m) = 0. \quad (\dot{\theta}_m = \pm \dot{\sigma}_m)$$

If $|\dot{\theta}_a| \neq |\dot{\sigma}_a|$, this means that

$$|\theta_m| + |\sigma_m| = \frac{1}{2}\pi.$$

Thus, whether the component waves of a stable mode move in the same or in opposite directions, the sum of the distances they have moved from a non-tilting state to the state of equal speeds is equal to the distance between the non-tilting states (a quarter of a wave length). Thus, in the state of equal speeds the ratio in (9.12) becomes

$$(9.14) \quad \tan^2 \theta_m = \pm \dot{\theta}_a / \dot{\sigma}_a = \pm \dot{\sigma}_b / \dot{\theta}_b = \cot^2 \sigma_m. \quad (\dot{\theta}_m = \pm \dot{\sigma}_m)$$

Finally, the value of the equal phase speeds is obtained from the phase equation (3.6), namely,

$$\dot{\theta}_m = \dot{\theta}_a \cos^2 \theta_m + \dot{\theta}_b \sin^2 \theta_m = (\dot{\theta}_a - \dot{\theta}_b) \cos^2 \theta_m + \dot{\theta}_b,$$

with the value of $\cos \theta_m$ substituted from (9.13), which gives

$$(9.15) \quad \dot{\theta}_m = \frac{\pm \dot{\theta}_a \dot{\sigma}_a - n^2}{\dot{\theta}_a \pm \dot{\sigma}_a} = \frac{\pm \dot{\theta}_b \dot{\sigma}_b - n^2}{\dot{\theta}_b \pm \dot{\sigma}_b} = \pm \dot{\sigma}_m.$$

These formulas define the state of equal speeds of the stable mode in terms of its phase velocities in the non-tilting states.

The symmetric waves in the statically stable shear layer have an infinite number of non-tilting states depending upon the value of the amplitude ratio of the component waves, which may be assigned arbitrarily. To construct an arbitrary non-tilting state as a resultant of normal modes, we need two pairs of modes. The frequency equation for this system must therefore be a quadratic equation in n^2 . We know already that the equation has two distinct roots, $n^2 = 0$, for the wave lengths of the stationary a-wave and the stationary b-wave

in (8.4), so it must have the form

$$[n^2 - 1 + n_a(1 + \mu)][n^2 - 1 + n_b(1 + \mu)] = Bn^2.$$

The value of B is found from the two limiting systems

$$U \rightarrow 0: \quad (n^2/\mu + n_a)(n^2/\mu + n_b) = (n^2/\mu)(B/\mu) = 0,$$

$$s \rightarrow 0: \quad n^4 - n^2(B - n_a - n_b + 2) + (n_a - 1)(n_b - 1) = 0.$$

The first of these gives the frequencies $-n^2 = m^2 = \mu n_{a,b}$ of the two stable modes in the resting three-layer system, and shows that B is independent of μ . One mode in the statically neutral shear layer ($s = 0$) is the Rayleigh mode (see 7.14) with $n^2 = -(n_a - 1)(n_b - 1)$. The second mode has therefore $n^2 = -1$. It is stable with the frequency $m = 1$ and the amplitude ratio $r = -1$. The shear vorticity at the deformed boundaries is everywhere neutralized by an equal amount of sliding vorticity along the boundaries. The mode is a neutral wave which moves downstream with the fluid. The coefficient of $-n^2$ in the frequency equation is the sum of the roots, so B has the value

$$B = n_a + n_b - 3 - (n_a - 1)(n_b - 1) = -(n_a - 2)(n_b - 2).$$

With this value substituted, the frequency equation takes the final form

$$(9.16) \quad [n^2 - 1 + n_a(1 + \mu)][n^2 - 1 + n_b(1 + \mu)] = -n^2(n_a - 2)(n_b - 2).$$

$n^2 = 0$ on the a-line and the b-line in the κ, μ -stability diagram (fig. 8) and for no other wave lengths, so only on these lines can n^2 change sign. The sign is indeed changed for one of the n^2 -roots when these lines are crossed, for we have seen that the a-line near the origin marks the transition from long stable to short unstable Helmholtz waves, and the b-line marks the transition (for $\mu = 0$) from the long unstable to the short stable Rayleigh waves. It follows that one family of modes is unstable in the spectral interval between these lines and stable in the rest of the spectrum. The

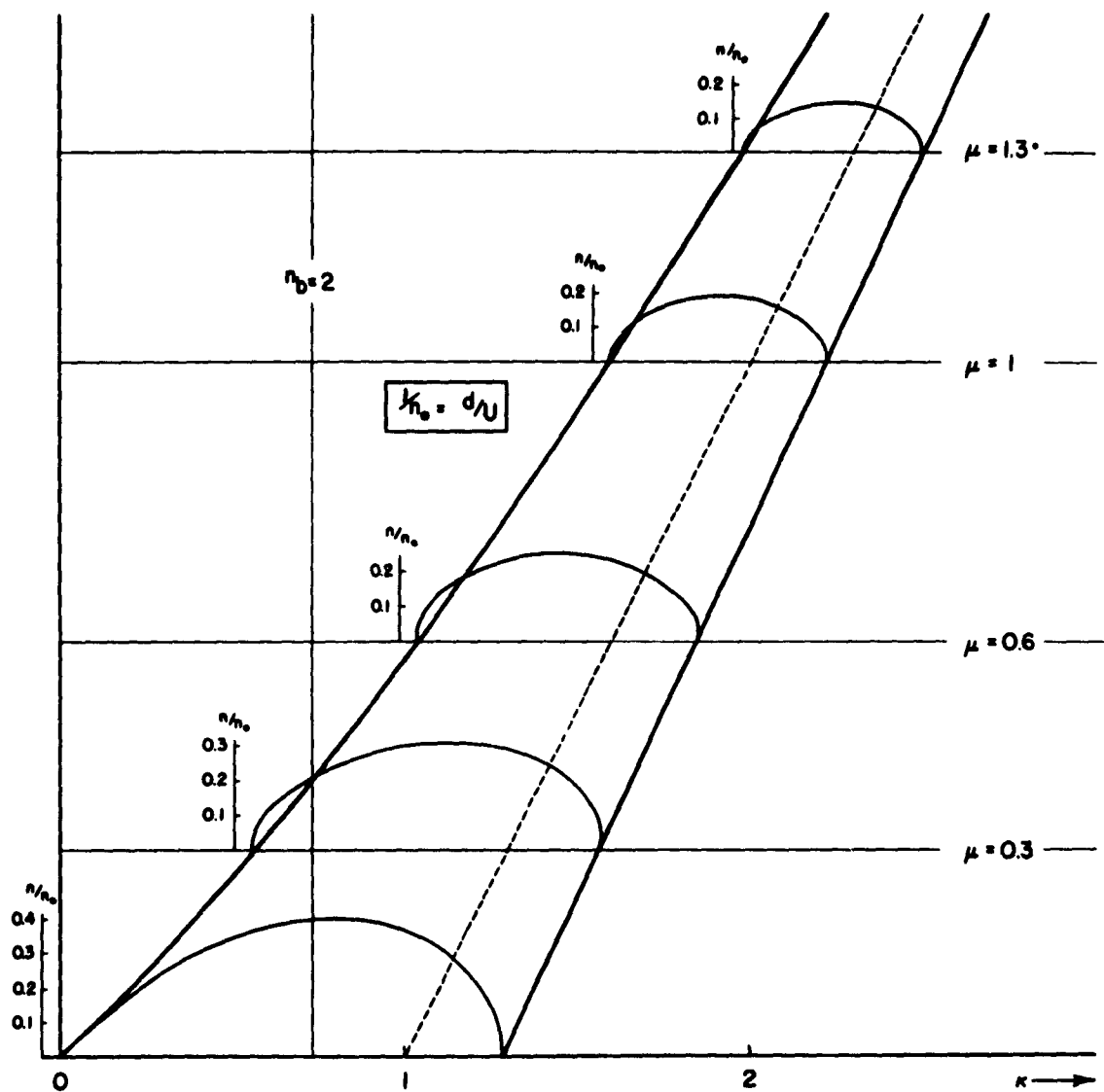


Fig. 9 Growth rate of the unstable mode

other family is stable. The growth rate of the unstable modes in units of U/d is shown for selected values of the Richardson number in fig. 9.

With the growth rate (or frequency) of the mode known, its non-tilting amplitude ratios are obtained from the products of the a- and b-equations in (9.9), namely,

$$(9.17) \quad \begin{aligned} \dot{\theta}_a \dot{\theta}_b &= (1 - u/r_a)(1 - u/r_b) = -n^2 \\ \dot{\phi}_a \dot{\phi}_b &= [1 - n_a(1 + r_a)][1 - n_b(1 + r_b)] = -n^2 \end{aligned}$$

The non-tilting amplitude ratios in turn determine the non-tilting phase velocities of the mode. These finally determine the other characteristic features of the mode. All the properties of the normal modes are therefore known in principle from their growth rate (or frequency). An arbitrary disturbance may be represented as the resultant of normal modes, so its evolution is known in principle as the resultant evolution of the modes.

To become familiar with the physical mechanism of this evolution we shall examine it in a wave where the mathematical theory is very simple, namely, the $(n_b = 2)$ -wave.

10. The $(n_b = 2)$ -wave. - This wave is a little longer than the wave of maximum growth rate in the statically neutral shear layer (7.16). It has the wave number

$$(10.1) \quad n_b = 2: \quad \kappa = \frac{1}{2}(1 + e^{-\kappa}) = 0.74 \quad (n_a = 2/\kappa - 2 = 0.7).$$

The value of n_a is closer to 0.71, but the slightly smaller value in (10.1) is accurate enough for a rough estimate.

The frequency equation (9.16) gives the growth rates for the two $(n_b = 2)$ -modes, namely,

$$(10.2) \quad \begin{aligned} (i): \quad n^2 &= 1 - n_a(1 + \mu) = 0.3 - \mu n_a, \\ (ii): \quad n^2 &= 1 - n_b(1 + \mu) = -(1 + 2\mu). \end{aligned}$$

The first mode is unstable below the a-line, and stable for greater static stability. The second is stable. Let us examine the evolution of these modes from their non-tilting states:

(i) The unstable mode: The non-tilting amplitude ratios of this mode are obtained from (9.17), namely

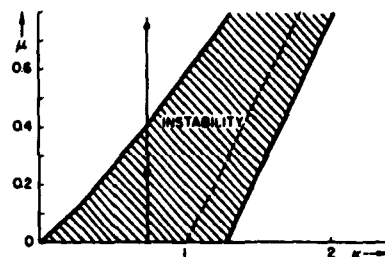
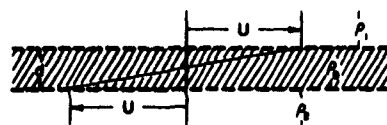
$$1 - n_a(1 + \mu) = -[1 - n_a(1 + r_a)][1 - 2(1 + r_b)] = -(1 - \mu/r_a)(1 - \mu/r_b).$$

These are satisfied by $r_a = \mu$, $r_b = 0$. The phase velocities in the non-tilting states are accordingly

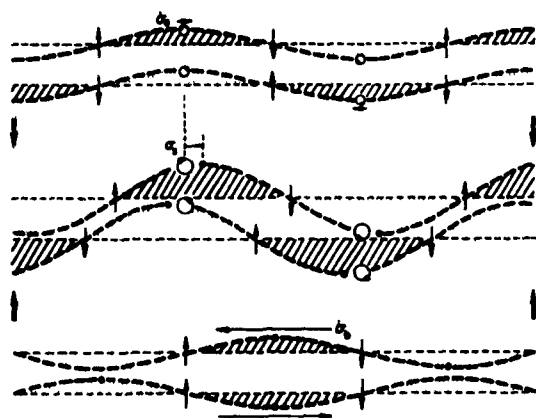
$$(10.3) \quad \begin{array}{lll} r_a = \mu: & \dot{\theta}_a = 0, & \dot{\sigma}_a = 0.3 - \mu n_a, \\ r_b = 0: & \dot{\theta}_b = \infty, & \dot{\sigma}_b = -1. \end{array}$$

The gravity wave remains permanently stationary in the non-tilting a-state and is absent from the b-state. The shear wave behaves formally like a Helmholtz wave (see 2.8), but the working mechanism is different. The Helmholtz wave moves upstream through the b-state with unit speed due to the kinematic constraint at the interface nodes. The shear wave in (10.3) moves upstream through the b-state with unit speed because it is here a simple Rayleigh wave with the sliding vorticities momentarily absent. In the a-state the mechanism is more similar: In the absence of static stability the Helmholtz wave moves downstream through the a-state with unit speed due to the sliding vorticity convergence at the downstream nodes. The statically neutral shear wave moves downstream with the speed 0.3, resulting in inflation of the shear layer at the downstream nodes. The static stability (and the sliding vorticity it excites) counteracts the downstream propagation of both waves, and becomes the dominating factor when sufficiently strong. The entire evolution of the shear wave in (10.3) may be represented graphically by a diagram very like the Helmholtz diagram in fig. 2, with straight $\dot{\sigma}_a$ - and $\dot{\sigma}_b$ -lines.

The shear wave has a stationary a-state for $\mu = n_a^{-1} - 1 = 0.4$, where the $(n_b = 2)$ -line meets the a-line (see fig. 10). For weaker static stability



$$\mu = 1/4$$



$$\mu = 3/4$$

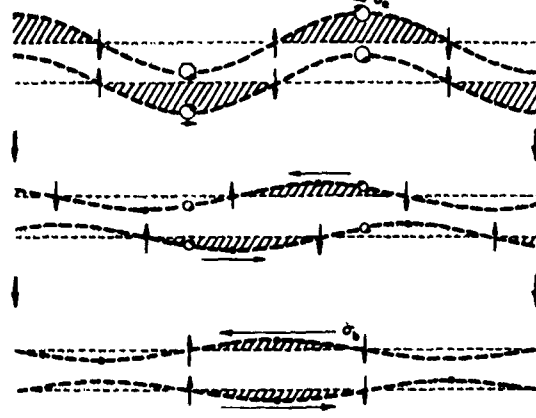


Fig.10 The unstable ($n_s=2$) - mode

the shear vorticity dominates, and the shear wave moves downstream through the a-state. As it leaves the a-state it begins to grow but slower than the statically neutral Rayleigh wave, because the sliding vorticity field (which remains behind in a growing a-state) retards the growth. As the static stability is increased from zero, it acts in two ways to reduce the growth of the shear-wave: (i) It reduces the downstream speed through the a-state and therefore gives the shear wave a stationary phase closer to the a-state which is less favorable for efficient growth. (ii) It retards the growth directly because the crest of the shear wave is downstream from the sliding vorticity maximum (see fig. 10).

For $\mu > 0.4$ the sliding vorticities are strong enough to move the shear wave upstream through the a-state. It is a stable wave which moves progressively upstream, while the gravity wave performs standing oscillations in the a-state. The shear vorticity and the sliding vorticity act on the changes of the interface amplitude in opposite sense in these waves. If $\mu = 1.85$ the two actions are equal, and the shear wave is a neutral wave which moves upstream with constant unit speed and constant amplitude. For weaker static stabilities (in the interval $0.4 < \mu < 1.85$) the shear vorticity dominates, so the shear wave decays and speeds up from the a-state to the b-state, and it slows down again and grows as the wave moves on to the next a-state. For greater static stability the sliding vorticities from the overweight dominates, and the shear wave changes speed and amplitude in opposite rhythm.

(ii) The stable mode: The amplitude ratios in the non-tilting states of this mode are given by the equations

$$1 + 2\mu = [1 - n_a(1 + r_a)][1 - 2(1 + r_b)] = (1 - \mu/r_a)(1 - \mu/r_b),$$

which give $r_a = -1$, $r_b = -(1 + \mu)$. The mode is stable and moves progressively downstream, passing the non-tilting states with the extreme phase velocities

$$(10.4) \quad \begin{array}{lll} -r_a = 1: & \dot{\theta}_a = 1 + \mu, & \dot{\sigma}_a = 1, \\ -r_b = 1 + \mu: & \dot{\theta}_b = (1 + 2\mu)/(1 + \mu), & \dot{\sigma}_b = 1 + 2\mu. \end{array}$$

In the statically neutral shear layer this wave has the constant amplitude ratio $r = -1$. The sliding vorticities cancel the shear vorticities so both fields drift downstream with the fluid. In statically stable shear layers the mode has the same general behavior for all values of μ : The resultant field is zero in the a-state, so the shear wave drifts downstream with the fluid through the a-state, while the gravity wave moves downstream faster than the fluid, driven by the overweight at the downstream nodes. After having left the a-state the sliding vorticity maximum is downstream from the interface crest where it is augmented by the overweight, so the gravity wave grows and slows down. The shear-vorticity tends to augment the interface amplitude, but this tendency is more than compensated by the stronger opposite action of the sliding vorticity. The shear wave decays and speeds up after leaving the a-state.

As the gravity wave slows down and the shear wave speeds up they approach the state of equal speed with the gravity wave at the maximum distance downstream from the shear wave. The amplitude ratio has here the value $r_m = r_a r_b = \sqrt{1 + \mu}$. The phase of the component waves is given by

$$\tan^2 \theta_m = \cot^2 \sigma_m = \dot{\theta}_a / \dot{\sigma}_a = 1 + \mu, \quad \text{or} \quad \cos 2\sigma_m = -\cos 2\theta_m = \mu / (2 + \mu),$$

and the equal speed of the waves is $(2 + 3\mu)/(2 + \mu)$. The $(\mu = 2)$ -mode has a very simple state of equal speeds ($\theta_m = 60^\circ$, $\dot{\theta}_m = 2$, $r_m = \sqrt{3}$).

After the state of equal speeds is passed the shear wave has the faster speed. Both waves arrive together in the b-state, the shear wave with the maximum speed $1 + 2\mu$, the gravity wave with the minimum speed $(1 + 2\mu)/(1 + \mu)$.

The evolution from an arbitrary state of the $(n_b = 2)$ -wave may now be found as the resultant evolution of the normal modes. For example the evolu-

tion from an initial a-state with the interface amplitude $A_s = A_{sa}$ and no sliding vorticity along the interface ($A_a = 0$) may be represented as the resultant of the a-state of the unstable mode with the amplitude $A_{s1} = A_{sa}/(1 + \mu)$ and the a-state of the stable mode with the amplitude $A_{s2} = A_{sa}\mu/(1 + \mu)$. The shear wave component of this resultant wave moves downstream through the a-state as a simple Rayleigh wave with the speed $1 - n_a = 0.3$. The momentarily absent gravity wave is immediately excited with the negative vorticity maximum at the downstream nodes. In terms of the evolution of the modes the appearance of this sliding vorticity may be interpreted as the downstream displacement of the negative vorticity of the stable mode, while the positive vorticity of the unstable mode remains behind in the a-state.

11. The modes on the a- and b-lines. - All the mathematical formulas for the symmetric waves in the shear layer are symmetric in the subscripts a and b, so the formulas for the modes on the a- and b-lines are the same with the subscripts interchanged.

On the a-line the roots of the frequency equation (9.16) are

$$(11.1) \quad \begin{aligned} (i) \quad n^2 &= 0, & r_a &= u, & n_a(1 + \mu) &= 1 \\ (ii) \quad n^2 &= 1 - n_b(1 + \mu) - (n_a - 2)(n_b - 2). \end{aligned}$$

The roots on the b-line are obtained by interchanging a and b. Consider first the ($n^2 = 0$)-mode.

(i) The ($n^2 = 0$)-mode: On the a-line this mode has a stationary neutral a-state with the amplitude ratio $r_a = u$. If started from the b-state, the mode moves toward the stationary a-state and approaches this state with an asymptotic linear growth rate. The amplitude ratio for the b-state of this mode cannot be obtained from (9.17), which here gives $n^2 = \dot{\theta}_a = \dot{\sigma}_a = 0$. However, it will be shown in the next section (see 13.4) that the non-tilting phase velocities of

the gravity wave in a normal mode are

$$(11.2) \quad \dot{\theta}_a = \frac{n^2 - 1 + n_a(1 + \mu)}{n_a - 2}; \quad \dot{\theta}_b = \frac{n^2 - 1 + n_b(1 + \mu)}{n_b - 2}$$

These values satisfy the upper equation in (9.17), and they reduce to the values for the $(n_b = 2)$ -modes in section 10. With these values substituted, the equations (9.9) give the non-tilting amplitude ratios of the mode, namely,

$$(11.3) \quad \frac{r_a}{\mu} = \frac{2 - n_a}{1 + \mu n_a + n^2}; \quad \frac{r_b}{\mu} = \frac{2 - n_b}{1 + \mu n_b + n^2}.$$

The b-state amplitude ratio of the $(n^2 = 0)$ -mode on the a-line is

$$(11.4) \quad \frac{r_b}{\mu} = \frac{2 - n_b}{1 + \mu n_b} = 1 - \frac{n_b(1 + \mu) - 1}{1 + \mu n_b} = \frac{\dot{\theta}_b}{\dot{\theta}_a} = \frac{\tan \theta}{\tan \sigma}. \quad (\text{a-line})$$

Since $r_a = \mu$, this formula gives the b-state phase velocity ratio as well (see 9.11,12). The b-state amplitude ratio is zero in the $(n_b = 2)$ -mode. It is positive in the shorter modes, increasing to the asymptotic limit $r_b = \mu$ in the very short modes. It is negative in the longer modes, decreasing to the asymptotic limit $r_b = -1/2$ in the very long modes $(\mu n_b \rightarrow 1)$.

From (11.2) and (11.4) the b-state phase velocities in this mode are

$$(11.5) \quad \begin{aligned} -\dot{\theta}_b &= \frac{n_b(1 + \mu) - 1}{2 - n_b} = \frac{1 + \mu n_b - 1}{2 - n_b} = \frac{\mu}{r_b} - 1, \\ -\dot{\sigma}_b &= \frac{n_b(1 + \mu) - 1}{\mu n_b + 1} = 1 + \frac{n_b - 2}{\mu n_b + 1} = n_b(1 + r_b) - 1. \end{aligned} \quad (\text{a-line})$$

On the a-line $n_b(1 + \mu) > 1$, or $\mu n_b + 1 > 2 - n_b$, so the shear wave moves upstream in all the $(n^2 = 0)$ -modes on this line. It moves with unit speed in the $(n_b = 2)$ -mode where the sliding vorticity is absent from the b-state. In the longer modes its upstream speed is greater than unity, but less than the speed of the Rayleigh wave because it is retarded by the negative sliding vorticity. The gravity wave on the other hand moves downstream in the long modes, faster than the fluid, driven by the overweight at the downstream nodes. After leaving the a-state the gravity wave grows from the overweight at its

crests and slows down. Also the shear wave grows, but slower than the Rayleigh wave because the growth is retarded by the sliding vorticity. Both waves move on in opposite directions toward asymptotic approach to the stationary a-state which they never quite reach. The gravity wave retains a residual phase upstream and the shear wave a residual phase downstream from the a-state. Since the energy transport is constant, their growth continues indefinitely at a linear rate ($\dot{A} = A_b \dot{\theta}_b$ and $\dot{A}_S = -A_{Sb} \dot{\sigma}_b$).

In the shorter ($n_b < 2$)-modes both component waves move upstream through the b-state. The speed of the shear wave is less than unity, but greater than the speed of the Rayleigh wave due to the positive sliding vorticity. The gravity wave is driven upstream by the overweight at the upstream nodes. In modes near the ($n_b = 2$)-mode this component is very weak and moves very fast. In the shorter modes with greater static stability it is stronger and has a slower speed. In the ($n_b = 1$)-mode for example (see 7.9) the gravity wave has the b-state velocity $-\dot{\theta}_b = n_a^{-1} - 1 = 0.77$. Since $r_b < \mu$ in the short modes (see 11.4), the gravity wave is leading and the shear wave trailing all the way from the b-state to the asymptotic a-state. The gravity wave grows from the overweight at its crest. The shear wave grows at a slower rate than the Rayleigh wave because its growth is retarded by the sliding vorticity. In the asymptotic a-state the mode has the residual downstream phase ratio $\sigma/\theta = \mu/r_b > 1$. The waves have here the same relative position as in the longer modes, and the asymptotic linear growth $\dot{A} = -A_b \dot{\theta}_b$, $\dot{A}_S = -A_{Sb} \dot{\sigma}_b$.

On the b-line the ($n^2 = 0$)-mode has a stationary neutral b-state with the amplitude ratio $r_b = \mu$. From (11.4) its a-state amplitude ratio is

$$(11.6) \quad \frac{r_a}{\mu} = \frac{2 - n_a}{1 + \mu n_a} = 1 + \frac{1 - n_a(1 + \mu)}{1 + \mu n_a} = \frac{\dot{\sigma}_a}{\dot{\theta}_a} = \frac{\tan \sigma}{\tan \theta} > 1 \quad (\text{b-line})$$

Here $n_a(1 + \mu) < 1$, or $2 - n_a > 1 + \mu n_a$, so $r_a > \mu$ in this mode. From (11.5) its

a-state phase velocities are

$$\begin{aligned} \dot{\theta}_a &= \frac{1 - n_a(1 + \mu)}{2 - n_a} = 1 - \frac{1 + \mu n_a}{2 - n_a} = 1 - \frac{\mu}{r_a} > 0, \\ (11.7) \quad \dot{\sigma}_a &= \frac{1 - n_a(1 + \mu)}{1 + \mu n_a} = \frac{2 - n_a}{1 + \mu n_a} - 1 = 1 - n_a(1 + r_a) > 0. \end{aligned} \quad (\text{b-line})$$

Both components move downstream through the a-state with the shear wave leading and the gravity wave trailing. The shear wave moves slower than the fluid, but faster than the Rayleigh wave due to the positive sliding vorticity. The gravity wave moves intrinsically upwind with a speed less than unity ($r_a > \mu$) driven by the overweight at the upstream nodes. As it trails behind the shear wave, it grows from the overweight at its crests, while the growth of the shear wave is retarded by the sliding vorticity. In the asymptotic b-state the mode has the residual upstream phase ratio $(\frac{1}{2}\pi - \theta)/(\frac{1}{2}\pi - \sigma) = r_a/\mu > 1$, and the growth continues at a linear rate ($\dot{A} = A_a \dot{\theta}_a$, $\dot{A}_g = A_{ga} \dot{\sigma}_a$).

If the wave length is much shorter than the depth of the shear layer, the fields from the lower boundary have no appreciable effect at the upper boundary and vice versa. In the short modes on the asymptotic (a,b)-line the shear wave and the gravity wave are both stationary and neutral in an arbitrary tilting state and have the amplitude ratio μ .

The ($n^2 = 0$)-mode on the a- and b-lines has a state of equal speed ($\theta + \sigma = \frac{1}{2}\pi$) downstream from the a-state. The phase and speed of this state are given by (9.14) and (9.15).

The stable mode: On the a-line this mode is defined by the second n^2 -root in (11.1). With this value substituted (11.2) gives the b-state phase velocity of the gravity wave in this mode, namely,

$$\dot{\theta}_b = 2 - n_a = 1 - \mu/r_b, \quad \text{so} \quad \mu/r_b = n_a - 1 = -\mu/(1 + \mu).$$

Its b-state amplitude ratio and phase velocities are therefore

$$(11.8) \quad -r_b = 1 + \mu, \quad \dot{\theta}_b = (1 + 2\mu)/(1 + \mu), \quad \dot{\sigma}_b = 1 + \mu n_b. \quad (\text{a-line})$$

The amplitude ratio is the same as in the stable ($n_b = 2$)-mode (see 10.4). The two modes have a rather similar behavior. In both the component waves move downstream through the b-state faster than the fluid. In the present mode the phase velocity difference is

$$(11.9) \quad \dot{\sigma}_b - \dot{\theta}_b = \mu[n_b(1+\mu) - 1]/(1+\mu) > 0. \quad (\text{a-line})$$

Also here the shear wave is leading the gravity wave downstream from the b-state. It slows down and grows from the b-state to the a-state, speeds up and decays from the a-state to the b-state and so on, while the gravity wave changes speed and amplitude in opposite rhythm.

The a-state amplitude ratio and phase velocities are obtained from the formula (9.11), namely,

$$(11.10) \quad \frac{r_b}{r_a} = \frac{\dot{\sigma}_b}{\dot{\theta}_b} = \frac{1 + \mu n_b}{1 + 2\mu}(1 + \mu) = \frac{\dot{\theta}_a}{\dot{\sigma}_a} = \frac{\tan \theta}{\tan \sigma} > 0, \quad (\text{a-line})$$

which gives

$$(11.11) \quad -r_a = \frac{1 + 2\mu}{1 + n_b \mu}, \quad \dot{\theta}_a = 1 - \mu/r_a, \quad \dot{\sigma}_a = \frac{\mu - r_a}{\mu + 1}. \quad (\text{a-line})$$

The a-state amplitude ratio has the asymptotic lower limit $-r_a = 1/2$ in the very long modes ($n_b \mu \rightarrow 1$), increasing to one in the ($n_b = 2$)-mode and on to the asymptotic upper limit μ in the very short modes.

The behavior of the stable mode on the b-line is obtained from equations (11.8-11) by interchanging the subscripts. Also here the amplitude ratio is negative with $-r_a = 1 + \mu$ and $-r_b$ increasing from unity in the statically neutral shear layer to the asymptotic upper limit μ in the very short modes. The gravity wave therefore moves progressively downstream faster than the fluid, leading the shear wave from the a-state to the b-state:

$$\dot{\theta}_a - \dot{\sigma}_a = \mu[1 - n_a(1 + \mu)]/(1 + \mu) > 0. \quad (\text{b-line})$$

It grows and slows down from the a-state to the b-state, speeds up and decays during the next quarter period and so on, while the shear wave changes speed

and amplitude in opposite rhythm, precisely as in the stable mode on the a-line. The behavior of the gravity wave is clearly understood in terms of the overweight at its crests and nodes. The shear wave responds to the joint action of shear- and sliding vorticities, with the sliding vorticity dominating in all states of this mode.

The stable mode has the same general characteristics on the ($n_b = 2$)-line and on the a- and b-lines: It has a negative amplitude ratio with minimum absolute value in the a-state and maximum value ($-r_b \approx 1 + \mu$) in the b-state. Its component waves move progressively downstream. They move faster than the fluid in all modes except the shear wave near the a-state in the long ($n_b > 2$)-modes. The gravity wave is the leading component from the a-state to the b-state. It is the dominating component in all states of the stable mode. Its sliding vorticity governs both the propagation and the growth and decay of the shear wave, so it moves intrinsically downstream through the fluid with pulsations opposite to the action from its own field. It is plausible that the stable mode has the same behavior in all parts of the spectrum.

12. The unstable mode in the shear layer. - We have seen that the statically stable shear layer has an unstable mode in the spectral interval between the a- and b-lines in the κ, μ -diagram. Near the a-line this mode has an asymptotic linearly growing state with the shear wave a residual phase downstream from the a-state. Near the b-line the unstable mode has an asymptotic linearly growing state with the shear wave a residual phase upstream from the b-state. At intermediate points the unstable mode has an exponentially growing state of stationary phase with the phase of the shear wave downstream from the a-state increasing continuously from zero on

the a-line to 90° on the b-line. The gravity wave has the proper phase upstream from the shear wave to make it grow at the rate of the mode (see 9.7).

The unstable mode has a rather simple kinematic structure on the straight line which is the common asymptote of the a- and b-lines, namely,

$$(12.1) \quad 1 + \mu = \kappa: \quad n_a(1 + \mu) = 1 - \alpha; \quad n_b(1 + \mu) = 1 + \alpha.$$

On this line the non-tilting phase velocities of the gravity wave in the normal modes (11.2) are

$$(12.2) \quad \begin{aligned} \dot{\theta}_a &= (\alpha - n^2)/(2 - n_a) = n \tan \theta_S, \\ -\dot{\theta}_b &= (\alpha + n^2)/(2 - n_b) = n \cot \theta_S. \end{aligned}$$

Its stationary phase in the unstable mode is given by

$$(12.3) \quad \tan^2 \theta_S = \frac{(\alpha - n^2)(2 - n_b)}{(\alpha + n^2)(2 - n_a)} = \frac{\alpha[n^2 + 2\kappa - 1] - [\alpha^2 + n^2(2\kappa - 1)]}{\alpha[n^2 + 2\kappa - 1] + [\alpha^2 + n^2(2\kappa - 1)]},$$

where n^2 is the positive root of the frequency equation

$$(12.4) \quad n^4 - \alpha^2 + n^2(2 - n_a)(2 - n_b) = 0.$$

In shear layers with very weak static stability ($\mu \ll 1$) this wave has very nearly the wave number $\kappa = 1$, and its frequency equation is

$$\mu \ll 1: \quad n^4 - \alpha^2 + n^2(1 - \alpha^2) = (n^2 + 1)(n^2 - \alpha^2) = 0. \quad (\kappa = 1)$$

The unstable mode has here the growth rate $n = \alpha = e^{-1}$, and from (9.7) the amplitude ratio $r_S = \mu(1 + \alpha^2)^{-1/2} \ll 1$. The very weak gravity wave has, from (12.2), the stationary phase $\tan \theta_S = (1 - \alpha)/(1 + \alpha)$, or $\tan(45^\circ - \theta_S) = \alpha = n$. From (9.7) the stationary phase of the shear wave is therefore $\sigma_S = 45^\circ$. The very weak gravity wave is 15.2° upstream from the shear wave, but gives no significant contribution to its growth. The growth of the shear wave is caused by the shear field, centered at the opposite interface, which acts with its full strength α in this mode.

In shear layers with very strong static stability ($\mu \gg 1$) the mode on the a,b-asymptote is very much shorter than the depth of the shear layer

($\kappa \gg 1$), so the component fields are reduced to insignificant strength at the level of the opposite interface. The wave at each interface behaves as if the wave at the other interface were absent. The frequency equation (12.4) for these very short modes is

$$\mu \gg 1: \quad n^4 + 4n^2 = n^2(n^2 + 4) = 0. \quad (\kappa \gg 1)$$

One mode is a stable neutral wave with the amplitude ratio $r = \mu$ which moves intrinsically upstream through the fluid with unit speed, so it is stationary in the symmetric frame. The second mode is a stable neutral wave with the amplitude ratio $r = -\mu$ which moves intrinsically downstream through the fluid with unit speed, so it moves downstream with the speed and frequency $m = 2$ in the symmetric frame.

For shear layers with dominating static stability ($\mu > 1$) a good approximation for the positive root is obtained by writing the frequency equation (12.4) in the form

$$n^2 = \frac{\alpha^2}{(2 - n_a)(2 - n_b) + n^2} = \frac{\kappa^2}{[e^{\kappa}(2\kappa - 1)]^2 + (n\kappa e^{\kappa})^2 - 1}.$$

For $\kappa \geq 2$, the term $e^{\kappa}(2\kappa - 1) > 21$, so practically no error is made if the last term in the denominator (-1) is ignored. The approximation is even better when also the middle term is ignored, giving the growth rate

$$(12.5) \quad \mu \geq 1: \quad n \approx \kappa e^{-\kappa} / (2\kappa - 1). \quad (\kappa \geq 2)$$

Even in the shear layer with static stability and shear of equal strength ($\mu = 1$) this formula gives the growth rate of the unstable mode on the a, b-asymptote with an accuracy better than one per cent. From (12.3) the stationary phase of the gravity wave in this mode, to the same accuracy, is given by

$$(12.6) \quad \mu \geq 1: \quad \cos 2\theta_s = \alpha \frac{\kappa^2 + 2\kappa - 1}{(2\kappa - 1)^2}. \quad (\kappa \geq 2)$$

For strong static stabilities ($\mu \gg 1$) the growth rate is $n = \frac{1}{2}\alpha$, the gravity wave has the stationary phase $\theta_s = \frac{1}{2}(\pi - \frac{1}{2}\alpha)$, and from (9.7) the shear wave has

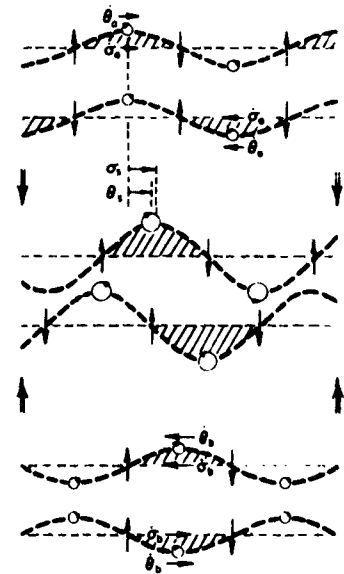
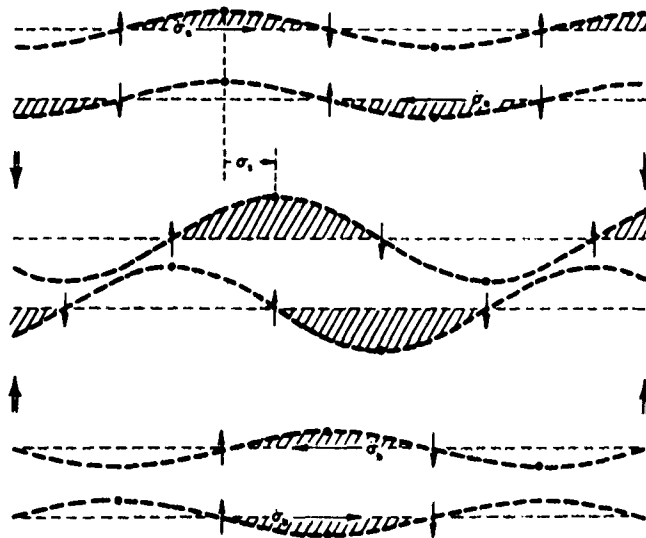
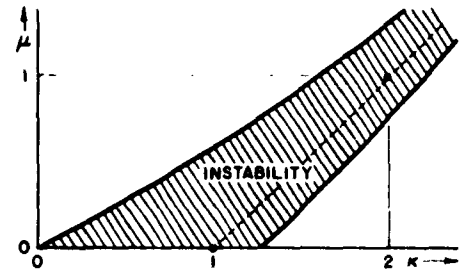
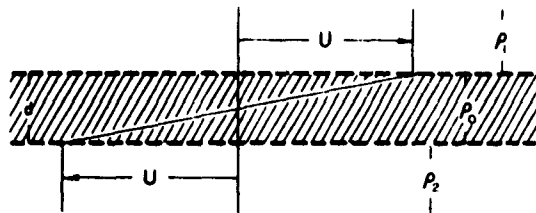


Fig.12 The unstable ($\mu = \kappa + 1$)-mode

the stationary phase $\sigma_s = \kappa(\pi + \frac{3}{8}\alpha)$. The amplitude ratio of the mode is $r_s = \mu$. With this stationary phase the shear wave would grow from its own field, if acting alone, at the rate α . The sliding vorticity of the very strong gravity wave with its maximum $\frac{1}{2}\alpha$ upstream from the interface crest reduces this growth rate by fifty per cent.

In conclusion we shall as a typical example exhibit the unstable ($\kappa=2$)-mode in the shear layer with static stability and shear of equal strength ($\mu=1$). It is shown in fig. 12. From (12.5) its growth rate is

$$\mu=1, \kappa=2: \quad n = 2/(3e^2) = 0.09 U\kappa = 0.18 U/d.$$

From (12.6) and (9.7) its component waves have the stationary phase

$$\cos 2\theta_s = 7/(9e^2) = 0.104, \quad \theta_s = 42^\circ, \quad \sigma_s = 47.2^\circ,$$

and the amplitude ratio $r_s = 0.99\%$. If the temperature difference of the outer layers is 5° Kelvin and their relative translation $2U = 10 \text{ ms}^{-1}$, the shear layer between them must have the depth $d = 600 \text{ m}$ to make it a ($\mu=1$)-layer (see 8.2). The ($\kappa=2$)-mode in this layer has the wave length 1880 m. In the asymptotic state of stationary phase its amplitude doubles every 7.7 minutes for as long as the linear theory is a valid approximation.

13. Shear layer between bounded outer layers. - To satisfy the kinematic conditions of zero vertical motion at the rigid boundaries of the outer layers (as in 4.1), the streamfunction of the field associated with the periodic vorticity field ($u^+ - u^-$) at the upper boundary of the shear layer has above and below that interface the values

$$\psi^+ = \psi_1 \text{sh } k(h-z)/\text{sh } kh, \quad (z > 0)$$

$$\psi^- = \psi_1 \text{sh } k(h+d+z)/\text{sh } k(h+d). \quad (z < 0)$$

Here h denotes the depth of the outer layers and d , as before, the depth of the shear layer. The tangential components of the field at the interface are

$$u^+ = \psi_1 k \cosh kh,$$

$$-u^- = \psi_1 k \cosh k(h+d).$$

The corresponding vorticity field at the interface is therefore

$$u^+ - u^- = (2\kappa/d)\psi_1.$$

Here we have introduced the non-dimensional parameter

$$(13.1) \quad \kappa = \frac{1}{2}kd[\cosh kh + \cosh k(h+d)].$$

At the lower boundary of the shear layer the field is reduced to the value $\alpha\psi_1$, where

$$(13.2) \quad \alpha = \sinh kh / \sinh k(h+d).$$

For waves which are much shorter than the depth of the outer layers these parameters reduce to the earlier unbounded layer values, namely,

$$kh \gg 1: \quad \kappa = kd, \quad \alpha = e^{-\kappa}.$$

From the dynamic condition in (7.2) the shear vorticity field associated with the deformation z_1 of the upper interface is

$$u^+ - u^- = (2U/d)z_1 = (2\kappa/d)\psi_1,$$

so the streamfunction of this field has the interface level value $(U/\kappa)z_1$, as in (7.3) for the unbounded system. The sliding vorticity field along the interface changes in accordance with the dynamic condition in (8.3), namely,

$$\frac{D}{Dt}(u^+ - u^-) = (2\kappa/d)\frac{D\psi_1}{Dt} = sg\frac{\partial z_1}{\partial x},$$

so the streamfunction of this field has the change

$$\frac{D\psi_1}{Dt} = (\mu/\kappa)\frac{\partial z_1}{\partial x}. \quad (\mu = \frac{1}{2}sgd/U^2)$$

Let us apply these conditions to an arbitrary state (9.1) of a symmetric wave. If the boundary conditions are satisfied at the upper interface for the symmetric wave, they are automatically satisfied at the lower interface, so it is sufficient to consider the upper conditions. The non-dimensional dynamic condition at the upper interface, namely,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \psi_1 = (\mu/\kappa) \frac{\partial z_1}{\partial x},$$

with the wave elements substituted from (9.1), becomes

$$Z - i\dot{Z} = (\mu/\kappa)\zeta. \quad (\text{Dynamic condition})$$

The kinematic condition at the upper interface, as in (7.12), is

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) z_1 = \frac{\partial}{\partial x} [\psi_1 + z_1/\kappa + \alpha(\psi_2 - z_2/\kappa)].$$

For the symmetric wave in (9.1) this condition becomes

$$\zeta - i\dot{\zeta} = Z + \zeta/\kappa - \alpha(Z^* + \zeta^*/\kappa). \quad (\text{Kinematic condition})$$

Substitute here the value of the shear wave parameter ζ from the dynamic condition. The resulting equation

$$\ddot{Z} - Z + (1 + \mu)(Z - \alpha Z^*)/\kappa = -2i\dot{Z} + i(\dot{Z} + \alpha\dot{Z}^*)/\kappa$$

has the real and imaginary parts

$$\begin{aligned} \ddot{X} - [1 - n_a(1 + \mu)]X &= (n_a - 2)\dot{Y}, & n_a &= (1 - \alpha)/\kappa \\ \ddot{Y} - [1 - n_b(1 + \mu)]Y &= -(n_b - 2)\dot{X}. & n_b &= (1 + \alpha)/\kappa \end{aligned} \quad (13.3)$$

They are the evolution equations for the gravity wave component of the symmetric wave.

In a state of stationary phase ($\theta = \theta_S = \text{const}$) only the amplitude of the wave changes with time, so the evolution equations (13.3) for such a state are

$$\begin{aligned} \ddot{A} - [1 - n_a(1 + \mu)]A &= (n_a - 2)\dot{A} \tan \theta_S, \\ \ddot{A} - [1 - n_b(1 + \mu)]A &= -(n_b - 2)\dot{A} \cot \theta_S. \end{aligned}$$

These are different forms of the same equation. It has solutions of the form $A \sim e^{nt}$ where the growth rate n is a root of the simultaneous quadratic equations

$$\begin{aligned} n^2 - 1 + n_a(1 + \mu) &= (n_a - 2)n \tan \theta_S = \dot{\theta}_a(n_a - 2), \\ n^2 - 1 + n_b(1 + \mu) &= -(n_b - 2)n \cot \theta_S = \dot{\theta}_b(n_b - 2). \end{aligned} \quad (13.4)$$

Elimination of θ_S gives the bi-quadratic growth-rate equation in (9.16),

which accordingly has the same form for bounded and for unbounded outer layers. The wave has a state of stationary phase θ_s with a real growth rate n in the spectral interval where the growth rate equation has a positive root $n^2 > 0$. The sum of the roots is never positive, so the growth rate equation has a positive root if the constant term is negative, that is in the spectral interval where

$$(13.5) \quad [n_s(1+u)-1][n_s(1+\mu)-1] < 0. \quad (\text{Instability})$$

This condition is satisfied in the spectral interval between the stationary neutral a-wave and the stationary neutral b-wave (8.4). In this interval the shear layer has one pair of unstable modes, one growing and one decaying at the rate n , where n is the real positive root of the growth rate equation. As in section 9 we shall regard this pair as the two states of stationary phase of a single mode.

14. The instability criterion for the bounded shear layer. - The condition for instability in (13.5) may be written

$$(14.1) \quad \kappa/(1-\alpha) > 1 + \mu > \kappa/(1+\alpha), \quad (\text{Instability})$$

where κ and α are the non-dimensional depth parameters in (13.1,2), namely,

$$(14.2) \quad \begin{aligned} \kappa &= \frac{1}{2}kd[\text{cth } kh + \text{cth } k(h+d)], \\ \alpha &= \text{sh } kh / \text{sh } k(h+d). \end{aligned}$$

For unbounded outer layers $\kappa = kd$, $\alpha = e^{-\kappa}$, and the condition gives instability in the spectral interval between the a- and b-lines in Goldstein's stability diagram in fig. 8. For bounded outer layers, choosing for the moment the length unit $\frac{1}{2}d = 1$, we have

$$\frac{\kappa}{k(1 \pm \alpha)} = \frac{\text{sh } k(h+2) \text{ch } kh + \text{sh } kh \text{ch } k(h+2)}{\text{sh } kh [\text{sh } k(h+2) \pm \text{sh } kh]},$$

so the lower limit in (14.1) is

$$\frac{\kappa}{k(1+\alpha)} = \frac{\text{sh } 2k(h+1)}{2 \text{ sh } kh \text{ sh } k(h+1) \text{ ch } k} = \frac{\text{ch } k(h+1)}{\text{sh } kh \text{ ch } k},$$

and the upper limit is

$$\frac{\kappa}{k(1-\alpha)} = \frac{\text{sh } 2k(h+1)}{2 \text{ sh } kh \text{ ch } k(h+1) \text{ sh } k} = \frac{\text{sh } k(h+1)}{\text{sh } kh \text{ sh } k}.$$

Substitution of these in (14.1) with arbitrary length units restored gives instability of the bounded shear layer in the interval

$$(14.3) \quad \text{cth } kh + \text{cth } \frac{1}{2}kd > 2(1+\mu)/kd > \text{cth } kh + \text{th } \frac{1}{2}kd. \quad (\text{Instability})$$

The upper limit is a stationary neutral a-wave and the lower limit is a stationary neutral b-wave.

If the depth of the shear layer shrinks to zero, no wave satisfies the lower b-wave limit. The vortex sheet has no stationary b-wave whether the layers it separates are bounded or not. However, as $kd \rightarrow 0$, the a-wave limit, using (8.2), may be written

$$\text{cth } kh - \text{sg}/kU^2 = (\frac{1}{2}kd)^{-1} - \text{cth } \frac{1}{2}kd \rightarrow -kd/6 \rightarrow 0,$$

so, as shown earlier in section 4, the vortex sheet between layers of finite depth is unstable for waves shorter than

$$(14.4) \quad \mu_h = (\text{sg}/kU^2) \text{th } kh < 1.$$

Here μ_h is the Richardson number of the Helmholtz wave in (4.5).

In the statically neutral ($\mu = 0$)-shear layer no wave satisfies the upper a-wave limit in (14.3). The condition for a stationary b-wave is here

$$\text{Stationary b-wave:} \quad \text{cth } kh = (\frac{1}{2}kd)^{-1} - \text{th } \frac{1}{2}kd. \quad (\mu = 0)$$

The longer modes are unstable and the shorter stable. This condition is represented graphically in a (kd, kh) -diagram by a line which leaves the origin with the slope $h = \frac{1}{2}d$ and has the vertical asymptote $kd = \kappa_g$ of the stationary b-wave in the shear layer between unbounded outer layers (see 7.9).

If the statically neutral shear layer fills more than one half the space between the outer rigid boundaries ($d > 2h$) it has no stationary b-wave and no unstable modes.

Waves which are very much longer than the total depth between the rigid boundaries, $k(d+2h) \ll 1$, obey the quasi-static approximation with good accuracy (see section 5). The instability condition (14.3) for the long quasi-static waves in the bounded shear layer may be written in the form

$$(14.5) \quad 1 > sgh/U^2 = \mu_h > 1 - 2h/d. \quad k(d+2h) \ll 1$$

It is determined by the Richardson number for the quasi-static Helmholtz waves (5.6) and the ratio between the depths of the shear layer and the outer layers. If $\mu_h > 1$ the shear layer has no unstable quasi-static waves. The $(\mu_h = 1)$ -shear layer has one quasi-static mode with a stationary neutral a-state and an asymptotic linearly growing a-state. The other mode is stable. If the shear layer occupies more than one half of the space between the outer boundaries ($d > 2h$) and its static stability has the value $\mu_h = 1 - 2h/d$, it has one quasi-static mode with a stationary neutral and a linearly growing b-state. Finally shear layers which occupy less than one half of the space between the outer boundaries have quasi-static instability if $\mu_h < 1$.

15. The statically stable Th-shear layer. - We shall now examine a quasi-symmetric shear layer which has continuous variations of velocity and density toward asymptotic values far above and below the central level, namely,

$$(15.1) \quad U = U_0 \tanh(2z/d), \quad \rho = \rho_0 \exp[-s \tanh^3(2z/d)].$$

We shall call this a Th-shear layer. It resembles the discontinuous Taylor-Goldstein constant-shear layer in section 8, and we shall show that it exhibits similar dynamic characteristics. Far from the central level the fluid moves in opposite directions with the same constant speed. The fluid at the central level has the maximum vorticity $2U_0/d$, so d is the depth of

the "equivalent" constant-shear layer. The Th-velocity profile in (15.1) is very close to the error-profile which develops by vorticity diffusion from a statically neutral vortex sheet (see fig. 6).

The Th-shear layer in (15.1) has no static stability at the central level. The variation of the density from far below to far above the central level is a small fraction of the density at the central level, so the layer is quasi-symmetric. Let the subscripts 1 and 2 denote the asymptotic limiting densities above and below the central level. They have the values

$$\begin{aligned}\rho_1 &= \rho_0 e^{-s} = \rho_0 (1 - s), \\ \rho_2 &= \rho_0 e^s = \rho_0 (1 + s),\end{aligned}\quad (s \ll 1)$$

so the stability parameter s has the same meaning as in the constant shear layer (see 8.1), namely,

$$(15.2) \quad s = (\rho_2 - \rho_1) / (\rho_2 + \rho_1) \ll 1.$$

In the following we shall use units of length and time such that $\frac{1}{2}d = U_0 = 1$, and introduce the abbreviated notations

$$\begin{aligned}T &= \text{th}(2z/d) = \text{th } z, \\ S &= \text{sech}(2z/d) = \text{sech } z.\end{aligned}\quad (d = 2)$$

These are connected by the relations

$$T' = S^2 = 1 - T^2, \quad T'' = 2SS' = -2TT' = -2TS^3.$$

Here and in the following the accent denotes differentiation with reference to height.

The static stability of the density stratification in (15.1) is represented by the logarithmic density gradient, namely,

$$(15.3) \quad \sigma = -(\ln \rho)' = (\ln \alpha)' = 3sT^2S^2.$$

We note that the stratified Th-shear layer in (15.1) has maximum static stability and maximum vorticity gradient at the levels

$$\begin{aligned}\sigma' &= 0: & T^2 &= \frac{1}{2}, & (z = 0.44d) \\ U''' &= 0: & T^2 &= \frac{1}{6}, & (z = 0.33d)\end{aligned}$$

Both lie inside the boundaries of the equivalent constant-shear layer ($z = \frac{1}{2}d$). We must expect some difference between the dynamic behavior of the Th-shear layer and the constant-shear layer on this account.

16. The wave equations for the Th-shear layer. - We now introduce into the Th-shear layer an arbitrary two-dimensional small amplitude wave disturbance in the planes of the basic flow whose velocity field $\underline{v}(x, z)$ is represented by a streamfunction as in (1.2). The resultant motion $\underline{V} = \underline{U}_1 + \underline{v}$ satisfies the dynamic equation

$$\frac{D\underline{V}}{Dt} = \underline{g} - (\alpha + \bar{\alpha})\nabla(p + \bar{p}),$$

where α and p denote the undisturbed static fields when the wave is absent, and the barred symbols denote the local variations of these fields due to the wave. We linearize this equation, substitute for the static pressure gradient from the static equation, $\nabla p = \underline{g}/\alpha$, and eliminate the dynamic pressure \bar{p} by vectorial differentiation. The result is

$$0 = \nabla \times \rho (D\underline{V}/Dt + \underline{g}\bar{\alpha}/\alpha) \approx \nabla \times (D\underline{V}/Dt + \underline{g}\bar{\alpha}/\alpha).$$

The approximation indicated in the last member on the right is the quasi-symmetry approximation which ignores the kinematic effect of the density gradient. (See section 4 and also the theory of the isothermal atmosphere in Chapter II, section 7.)

For the two-dimensional incompressible flow the inertial term in the vorticity equation may be transformed as follows

$$\nabla \times \frac{D\underline{V}}{Dt} = \nabla \times \left(\frac{D\underline{V}}{Dt} + \underline{Q} \times \underline{V} \right) = \frac{D\underline{Q}}{Dt}. \quad (\underline{Q} = \nabla \times \underline{V} = U' \underline{j} + \nabla \times \underline{v})$$

The fluid is incompressible so the local change of the specific volume comes from the vertical displacement z_g of the fluid particles from their

undisturbed levels. Its value to the linear approximation, using the notation in (15.3), is therefore

$$\bar{\alpha}/\alpha = -(\ln \alpha)' z_s = -\sigma z_s.$$

With these substituted, the quasi-symmetric vorticity equation above takes the form

$$(16.1) \quad j \cdot \tilde{\omega} \times \frac{D\tilde{y}}{Dt} = \frac{D}{Dt} (U' + q) = \sigma g \frac{\partial z_s}{\partial x},$$

where

$$(16.2) \quad q = j \cdot \tilde{\omega} \times \tilde{y} = -\nabla^2 \psi = q_s + q_g$$

is the vorticity field of the wave. The vorticity equation (16.1) is the dynamic equivalent of the corresponding condition at a statically stable interface between fluid layers of different densities (see 2.4), and it represents the same physical principle: The vorticity of the fluid particle is changed in the sense of its overweight which is proportional to the static stability and the slope of the isosteres. The vorticity is the sum of the shear vorticity of the basic flow U' and the vorticity which is added by the wave motion. As in the constant shear layer the added periodic vorticity field q in the wave may be regarded as the sum of two component fields as indicated in (16.2). One component q_s comes from the vertical displacement of the shear vorticity U' of the basic flow. We call this component the shear wave. The second component q_g is the part of the vorticity which is changed by the overweight and will be called the gravity wave. In a statically neutral shear layer ($\sigma = 0$) the gravity wave is absent, and the dynamic equation (16.1) gives the shear vorticity change

$$(16.3) \quad \frac{Dq_s}{Dt} = -\frac{DU'}{Dt} = -U''w = -U'' \frac{Dz_s}{Dt},$$

or, when integrated, the shear vorticity itself

$$(16.4) \quad q_s = -U'' z_s.$$

It is positive (in the sense of the basic shear) for displacements away from the central level, and negative for displacements toward that level. Evidently the same relation (16.4) holds for the shear vorticity in the statically stable shear layer so, with (16.3) substituted, the dynamic vorticity equation (16.1) gives the change of the vorticity field in the gravity wave, namely,

$$(16.5) \quad \frac{D}{Dt} (q - q_s) = \frac{Dq_s}{Dt} = \sigma g \frac{\partial z_s}{\partial x}.$$

Let us now assume that the shear wave and the gravity wave are in phase (or in opposite phase) at all levels, with the amplitude ratio

$$(16.6) \quad r(z) = q_g/q_s. \quad (\text{Equal phase})$$

However, the wave may have an arbitrary tilt. In such a state of the wave the extreme values of the vorticity coincide at all levels with the crests and troughs of the shear wave, so the gravity wave is momentarily neutral in a state of equal phase. Let $G(z)$ denote the intrinsic upstream speed of the gravity wave through the fluid in a state of equal phase. In a frame which moves locally with the gravity wave the fluid moves downstream with the speed G and the gravity wave has at the moment no local change, so (16.5) for a state of equal phase becomes

$$G \frac{\partial q_g}{\partial x} = \sigma g \frac{\partial z_s}{\partial x} = - \frac{\sigma g}{U''} \frac{\partial q_s}{\partial x},$$

which gives the intrinsic upstream phase velocity

$$(16.7) \quad \frac{G}{U} = - \frac{\sigma g}{UU''} / r = 1.5\mu/r. \quad (\mu = \frac{1}{2} \frac{sgd}{U_0^2})$$

In the last term on the right we have substituted from (15.1-3)

$$- \frac{\sigma g}{UU''} = \frac{3sgT^2T'}{2T^3T'} = 1.5sg = 1.5\mu,$$

so the Richardson number μ has the same meaning as in the constant shear layer (see 8.2). The phase velocity formula for the gravity wave in (16.7)

resembles the corresponding formula (9.6) for the constant shear layer which, for a state of equal phase, gives the intrinsic upstream speed μ/r . However, the gravity wave in the Th-shear layer has 50 per cent more vorticity than a wave with the same relative speed in the constant-shear layer. The reason for this is at least partly that $U < U_0$ at all levels in the Th-shear layer, so the gravity wave needs less overweight to have the same relative speed.

To derive a corresponding phase velocity formula for the shear wave we must specialize further to a non-tilting state of equal phase where also the shear wave is instantaneously neutral. Let $C(z)$ denote the upwind intrinsic speed of the shear wave in a non-tilting state. The kinematic condition for such a state may be written

$$\frac{Dz_S}{Dt} = C \frac{\partial z_S}{\partial x} = \frac{\partial \psi}{\partial x},$$

which gives the intrinsic upwind phase velocity

$$(16.8) \quad \frac{C}{U} = \frac{\psi}{Uz_S} = - \frac{U''}{U} \frac{\psi}{q_S} = - (1+r) \frac{U''\psi}{Uq} = (1+r) \frac{2S^2\psi}{q}.$$

This formula resembles the corresponding formula for the constant-shear layer in (9.8), which gives the intrinsic upstream speed $(1+r)n_{a,b}$ for the shear wave, where the factors $n_{a,b}$ are constants. However the corresponding factor $2S^2\psi/q$ in (16.8) is in general a function of height. The Th-shear layer is therefore a more complex dynamic system.

The conditions for a stationary neutral non-tilting state of a wave in the Th-shear layer are from (16.7-8)

$$(16.9) \quad r = 1.5\mu, \quad q = -7^2\psi = (2+3\mu)S^2\psi. \quad (G = C = U)$$

The first condition makes the gravity wave stationary. The second condition makes the shear wave stationary. When the streamfunction has a sinusoidal variation in the x-direction this condition is a differential equation

of the hypergeometric type. It has been studied by R. V. Garcia* who was able to find all the stationary neutral waves in this Th-shear layer. We shall derive Garcia's (still unpublished) solutions of (16.9) in section 20 below. But first we shall consider a few simple waves which may be derived with less mathematical labor.

17. Some stationary waves in the Th-shear layer. - The form of the stationary wave equation in (16.9) makes it plausible that it may have solutions whose height variations are simple functions of T and S, say

$$(17.1) \quad \psi = A_n T^n S^m \cos kx.$$

For the moment m and n are arbitrary non-negative constants. From (16.9) the corresponding vorticity q of this wave is given by

$$(17.2) \quad q/\psi = k^2 - \psi''/\psi.$$

Logarithmic height differentiation of (17.1) gives

$$\psi'/\psi = nT'/T + mS'/S = n/T - (m+n)T,$$

and a second differentiation gives

$$\begin{aligned} \psi''/\psi &= [n/T - (m+n)T]' + [n/T - (m+n)T]^2 \\ &= -[(m+n)(m+n+1) - n(n-1)/T^2]S^2 + m^2. \end{aligned}$$

With this value substituted, (17.2) becomes

$$q/(S^2\psi) = (m+n)(m+n+1) - n(n-1)/T^2 + (k^2 - m^2)/S^2.$$

We shall only consider waves of this type for which $m=k$, so the last term on the right is zero. Introducing the abbreviation

$$(17.3) \quad K_n = (k+n)(k+n+1),$$

these waves have the resultant fields

$$\begin{aligned} (17.4) \quad \psi_n &= A_n T^n S^k \cos kx, \\ q_n &= A_n [K_n T^n - n(n-1)T^{n-2}] S^{k+2} \cos kx. \end{aligned}$$

* Garcia, R. V., 1956: Barotropic Waves in Straight Parallel Flow with Curved Velocity Profile. Tellus, 8, pp. 82-93.

The factor in (16.8) which determines the phase velocity of the shear wave component of these waves has the value

$$(17.5) \quad q/(S^2 \psi) = K_n - n(n-1)/T^2.$$

This factor is independent of height in the $(n=0)$ -wave and the $(n=1)$ -wave, so the instantaneous relative phase speeds of these non-tilting waves are independent of height if the amplitude ratio is the same at all levels. They have from (16.7-8) the values

$$(17.6) \quad G/U = 1.5\mu/r, \quad C/U = 2(1+r)/(K_0 \text{ or } K_1).$$

From (17.4) these families of waves have the resultant streamfunctions

$$(17.7) \quad \psi_0 = S^k \cos kx, \quad \psi_1 = TS^k \cos kx,$$

with an arbitrary constant amplitude factor omitted.

The $(n=0)$ -family is symmetric with the same phase above and below the central level. It is a b-state of a symmetric wave. The $(n=1)$ -family is symmetric with opposite phase above and below the central level. It is an a-state. If the amplitude ratio has the value 1.5μ the gravity wave is stationary in these waves. If, furthermore, K_0 has the value $2+3\mu$ also the shear wave is stationary in the $(n=0)$ -family. If K_1 has this value the shear wave is stationary in the $(n=1)$ -family. The conditions to be satisfied by these two families of stationary waves are accordingly

$$(17.8) \quad \text{stationary a-wave:} \quad r_a = 1.5\mu, \quad K_1 = v, \quad (v = 2 + 3\mu).$$

$$(17.9) \quad \text{stationary b-wave:} \quad r_b = 1.5\mu, \quad K_0 = v, \quad K_n = (k+n)(k+n+1)$$

These families represent two of the solutions of (16.9) which were found by Garcia. He was able to prove that these are the only stationary waves in the Th-shear layer (15.1) if its static stability is weak ($\mu < 4/3$). For stronger static stability there are other stationary waves (see section 20).

Let us now compare the a- and b-waves in (17.7) with the corresponding waves in the constant shear layer.

18. The a-wave (n=1). - This wave has from (17.5 and 7) an instantaneous a-state with the field

$$(18.1) \quad \psi = TS^k \cos kx, \quad q/(S^2 \psi) = K_1.$$

Its intrinsic upstream phase velocities for an arbitrary amplitude ratio are

$$(18.2) \quad G_a/U = 1.5\mu/r_a; \quad C_a/U = 2(1+r_a)/K_1.$$

It is a stationary neutral a-wave if the amplitude ratio has the value 1.5μ , and the parameter K_1 (see 17.3) has the value

$$(18.3) \quad K_1 = (k+1)(k+2) = 2+3\mu.$$

This condition is represented in a k, μ -diagram by a parabola through the origin.

To compare this parabola with Goldstein's a-line for the stationary a-waves in the constant-shear layer, we recall that $k = \frac{1}{2}kd = \frac{1}{2}\kappa$, so the stationary a-wave condition in (18.3) may be written

$$(18.4) \quad \mu = \frac{1}{2}\kappa + \kappa^2/12. \quad (\kappa = kd)$$

Near the origin this parabola coincides with Goldstein's a-line, $\mu = \kappa/(1 - e^{-\kappa}) - 1$ to the order of κ^2 . For shorter waves the parabola lies above the Goldstein line, but the two curves are almost coincident for $\mu < 1$ (see fig. 19). For example for $\kappa = 1$ the stationary a-wave in the constant-shear layer has the static stability $\mu = (e-1)^{-1} = 0.582$, while the same wave in the Th-shear layer has the static stability $\mu = 7/12 = 0.583$.

For other waves with the amplitude ratio 1.5μ the gravity wave is stationary while the shear wave moves intrinsically upstream with the speed C_a in (18.2). Its non-dimensional downstream speed with reference to the fluid at the central level is

$$\dot{\sigma}_a = (U - C_a)/U_0 = T[1 - (2+3\mu)/K_1]. \quad (\text{Th-shear})$$

Let us compare this speed at the distance $\frac{1}{2}d$ from the central level

($T = 0.76 \sim 3/4$) with the a-state speed in the constant shear layer, namely,

$$\dot{\sigma}_a = 1 - (\mu + 1)(1 - e^{-\mu})/\mu. \quad (\text{Constant-shear})$$

The ($\mu = 1$)-wave, for example, has here the a-state speed

$$\dot{\sigma}_a = \mu \left[1 - \frac{2 + 3\mu}{(\frac{1}{2} + 1)(\frac{1}{2} + 2)} \right] = \frac{7 - 12\mu}{20}, \quad (\text{Th-shear})$$

$$\dot{\sigma}_a = 1 - (\mu + 1)(1 - e^{-1}) = \frac{7.36 - 12.64\mu}{20}. \quad (\text{Constant-shear})$$

For longer waves the agreement is even closer.

This good agreement in the dynamic a-state behavior of the symmetric wave (18.1) in the Th-shear layer with the symmetric wave in the constant-shear layer suggests that the two layers have similar dynamic characteristics, at least for waves longer than about six shear layer depths.

19. The b-wave ($n = 0$). - This wave has from (17.5 and 7) an instantaneous b-state with the field

$$(19.1) \quad \psi = S^k \cos kx, \quad q/(S^2 \psi) = K_0 = k(k+1).$$

Its intrinsic upstream phase velocities for an arbitrary amplitude ratio are

$$G_b/U = 1.5\mu/r_b; \quad C_b/U = 2(1 + r_b)/K_0.$$

This wave is stationary if its amplitude ratio is 1.5μ and the parameter K_0 has the value

$$(19.2) \quad K_0 = k(k+1) = 2 + 3\mu,$$

which is represented in the k, μ -diagram by a parabola identical to the a-parabola in (18.3) only displaced unit distance toward increasing k (see fig. 19).

This parabola departs very much from the b-line for the constant shear layer. The reason is that the wave in (19.1) has the maximum vorticity near the central level where the wave in the constant shear layer has no added vorticity. The partial field from the vorticity in the central

region augments the upstream propagation of the shear wave at all levels, so the stationary b-wave in (19.1) is shorter than the stationary b-wave in the constant-shear layer. If the excess vorticity in the central region were removed from the wave, the shear wave would move downstream at all levels, similar to the same wave in the constant shear layer.

It is impossible to set up a stationary b-wave in the Th-shear layer unless it is given excess vorticity near the central level, where no such vorticities can be generated by kinematic advection or by dynamic action of the overweight. However, with the aid of the field of the a-wave in (18.1) we may construct a b-wave which has no excess vorticity at the central level simply by changing the sign of the a-wave vorticity in the lower layer. Accordingly this b-wave has the vorticity field

$$(19.3) \quad q = K_1 |T| S^{k+2} \cos kx.$$

Above the central level this wave has the same vorticity as the a-wave in (18.1). The partial field from the vorticities below the central level is Laplacean in the upper region for both waves, so the two waves differ only by a Laplacean field. Above the central level the streamfunction of the vorticity wave in (19.3) accordingly has the form

$$\psi = (TS^k + Ae^{-kz}) \cos kx. \quad (z > 0)$$

Since the field is symmetric with respect to the central level we have $\psi'(z=0) = 0$, so $A = k^{-1}$. The streamfunction of the vorticity wave (19.3) is therefore

$$(19.4) \quad \psi = (|T|S^k + k^{-1}e^{-k|z|}) \cos kx.$$

If this wave has the amplitude ratio 1.5μ , the gravity wave is stationary, and from (16.8) the shear wave in the upper region has the relative speed

$$(19.5) \quad \frac{C_b}{U} = \frac{3\mu + 2}{(k+1)(k+2)} \left(1 + \frac{1}{kT(1+T)^k}\right). \quad (r = 1.5\mu)$$

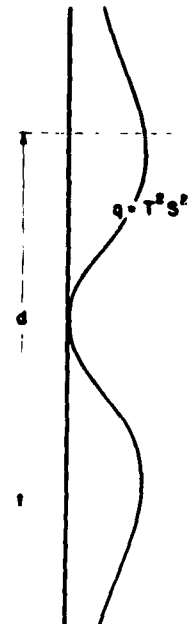
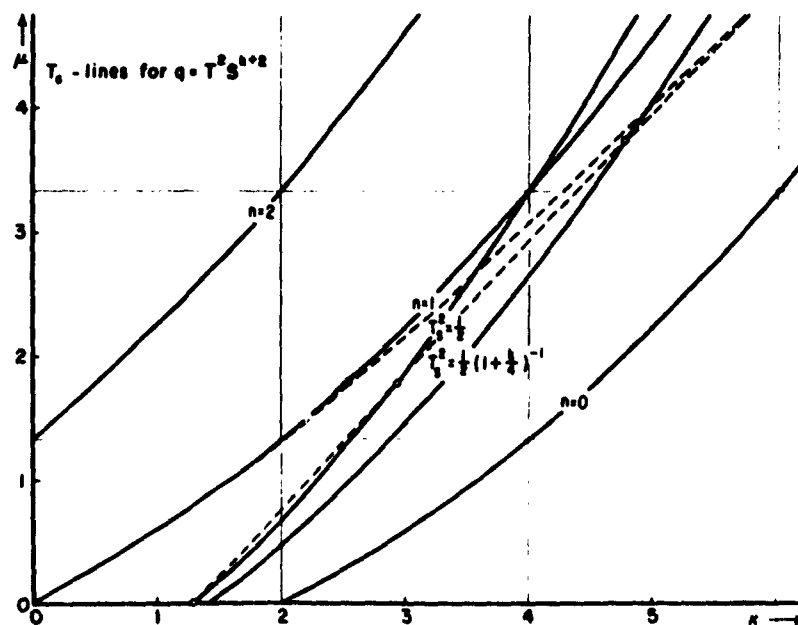
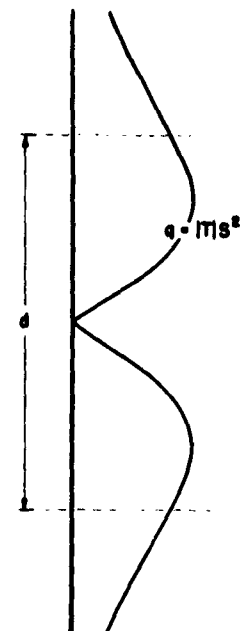
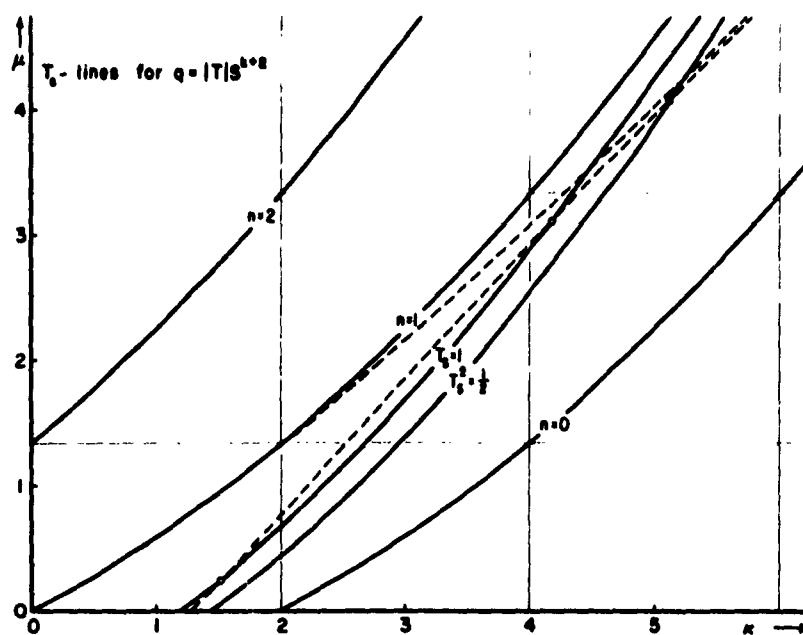


Fig. 19 Stationary waves in a Th-shear layer

Its upstream speed is greater than in the corresponding a-wave (18.2). For very short wave lengths it has the same asymptotic behavior as the a-wave. In particular, the condition for a stationary short wave is the same (18.3) for both waves. For the longer b-waves in (19.4), on the other hand, the relative speed of the shear wave is a function of height, so for no value of the static stability are these waves stationary at all levels simultaneously. The wave is stationary at a given level $z = z(T_S)$ if the static stability has the value of

$$(19.6) \quad 3\mu + 2 = (k+1)(k+2) \left(1 + \frac{1}{kT_S(1+T_S)^k} \right)^{-1}, \quad C_b(T_S) = U.$$

At higher levels this wave moves slower than the air, at lower levels faster. The wave is stationary far from the central level if

$$3\mu + 2 = (k+1)(k+2)[1 + (k2^k)^{-1}]^{-1}. \quad C_b(T_S=1) = U_0.$$

This line is shown in the upper μ, κ -diagram in fig. 19. It meets the Goldstein line for the stationary b-waves in the constant shear layer at $\kappa = 2k = 1.5$ and at $\kappa = 4.15$. In the spectral interval between these waves the stationary b-wave in the constant shear layer has a little more static stability. For example the $(\kappa = 2)$ -wave in the constant shear layer is stationary if $\mu = (e^2 - 1)/(e^2 + 1) = 0.76$, while the same wave in (19.4) is stationary far from the central level if $\mu = 2/3$.

We recall from (15.4) that the Th-shear layer has maximum static stability at the level of $T^2 = 1/2$. With this value of T_S substituted, equation (19.6) marks the waves in (19.4) which are stationary at the level of maximum stability. The corresponding line is shown in the upper diagram in fig. 19. These waves have considerably less static stability than the stationary b-waves in the constant shear layer. The main reason is probably that the vorticity field in (19.3) has unrealistically large

values near the center level where there is no physical mechanism for the generation of vorticity.

Let us therefore consider a third class of b-waves which has very little vorticity near the central level, namely waves with the vorticity distribution $q \sim T^2 S^{k+2}$. The streamfunction of this wave is obtained by taking the resultant of two of the waves in (17.4). Omitting the trigonometric factor, the waves of $n=0$, and $n=2$ are

$$q_0 = K_0 S^{k+2}; \quad \psi_0 = S^k \quad (2)$$

$$q_2 = (K_2 T^2 - 2) S^{k+2}; \quad \psi_2 = T^2 S^k. \quad (K_0)$$

The resultant of these, augmented by the factors to the right, is

$$(19.7) \quad q = K_0 K_2 T^2 S^{k+2}; \quad \psi = (2 + K_0 T^2) S^k.$$

If the amplitude ratio of this wave is 1.5μ , its gravity wave is stationary and its shear wave moves with the intrinsic upstream speed

$$(19.8) \quad \frac{C_b}{U} = \frac{3\mu + 2}{(k+2)(k+3)} \left(1 + \frac{2}{k(k+1)T^2}\right). \quad (r = 1.5\mu)$$

This wave has maximum vorticity at the levels of $T^2 = \frac{1}{2}(1 + \frac{1}{2}k)^{-1}$. For the very long waves these coincide with the levels of maximum stability. For $k=2$ they coincide with the levels of maximum vorticity gradient. For $k=0.8$ the levels of maximum vorticity are in the middle between the levels of maximum stability and maximum vorticity gradient. The wave in (19.7) is stationary at the level of maximum stability ($T^2 = \frac{1}{2}$) if the stability has the value of

$$(19.9) \quad 3\mu + 2 = \frac{k(k+1)(k+2)(k+3)}{k^2 + k + 4} \quad C_b(T^2 = \frac{1}{2}) = U$$

In a μ, k -diagram (see lower diagram in fig. 19) this line meets the b-line for the constant shear layer at the points $(\mu=0, \kappa \approx \kappa_g = 1.278...)$ and $(\mu=1.75, \kappa=2.9)$. Between these points the wave in (19.9) has a

little less static stability, but otherwise its properties are quite similar to those of the stationary b-wave in the constant shear layer. Like the stationary a-waves in section 18 this similarity probably reflects a fundamental dynamic similarity between the Th-shear layer and the constant shear layer. However, whereas the constant shear layer has only these two families of stationary waves, the Th-shear layer has an infinite number of such families.

20. The stationary waves in the Th-shear layer. (Garcia's solutions.) - From (16.9) the general condition for a stationary wave in the Th-shear layer with reference to the fluid at the central level ($G = U = C$) is

$$(20.1) \quad q = -\nabla^2 \psi = v S^2 \psi. \quad (v = 3\mu + 2)$$

When the streamfunction has a sinusoidal variation in the x-direction, Garcia noticed that this differential equation, by suitable transformation, is the hyper-geometric differential equation. As such its solutions may be represented by the series

$$(20.2) \quad \psi = \sum_{m=0}^{m=n} A_m T^m S^k \cos kx,$$

where the coefficients A_m are the coefficients of the hyper-geometric series. These coefficients are in the present case easily determined by the formulas in (17.4): The field in (20.2) has the vorticity

$$(20.3) \quad q = \sum_{m=0}^{m=n} A_m [K_m T^m - m(m-1)T^{m-2}] S^{k+2} \cos kx = \sum_{m=0}^{m=n} v A_m T^m S^{k+2} \cos kx.$$

In a stationary wave this vorticity must have the value in (20.1), as indicated. The coefficient for T^m gives

$$A_m (K_m - v) = (m+1)(m+2)A_{m+2}.$$

Repeated use of this recursion formula gives

$$(20.4) \quad A_m = A_0 (K_0 - \nu)(K_2 - \nu) \dots (K_{m-2} - \nu)(m!)^{-1}, \quad (m \text{ even})$$

$$(20.5) \quad A_m = A_1 (K_1 - \nu)(K_3 - \nu) \dots (K_{m-1} - \nu)(m!)^{-1}. \quad (m \text{ odd})$$

Thus ψ in (20.2) is a solution of the differential equation if

$$(20.6) \quad \nu = K_n, \quad \text{or} \quad 3\mu + 2 = (k+n)(k+n+1),$$

where n is any non-negative integer. If n is an even integer, $A_1 = 0$ and the coefficients A_m have the values in (20.4). If n is an odd integer, $A_0 = 0$ and the coefficients A_m have the values in (20.5).

The stationary wave conditions in (20.6) are represented in the k, μ -diagram by a family of identical parabolas, each displaced unit distance toward decreasing k from the next lower member. The $(n=0)$ -parabola marks the stationary b-waves (see 19.2) which have no nodal plane. The $(n=1)$ -parabola marks the stationary a-waves (see 18.3) which have one nodal plane at the central level. The $(n=2)$ -parabola marks a family of stationary waves with two nodal planes and so on. If n is odd, the central level is a nodal plane and the field has opposite phase at equal heights above and below. The wave is of the a-type. If n is even, the wave has the same phase and amplitude at equal heights above and below the interface. The wave is of the b-type.

By examining all the solutions of the hyper-geometric equation Garcia found that the solutions in (20.6) represent all the stationary waves in the Th-shear layer in (15.1) which have bounded and continuous velocity fields.

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